Sentiments and Real Business Cycles*

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Abstract

We introduce sentiments under incomplete information into an otherwise standard real business cycle model. Individual firms receive signals about their idiosyncratic demand shocks which are confounded by sentiments. Sentiments coordinate optimal decisions of individuals through their extraction of the aggregate economic conditions from the signals. We show that there exists a sentiment-driven rational expectations equilibrium in addition to a fundamental equilibrium. Optimistic sentiments boost the aggregate economy, leading to positive comovements among output, consumption, investment, and hours worked. We calibrate a full-blown dynamic stochastic general equilibrium model based on U.S. aggregate data and find that sentiment shocks substantially amplify the aggregate fluctuations.

Keywords: sentiments, real business cycles, self-fulfilling equilibria, business cycle comovement
JEL Classification: E20, E32

1. Introduction

There is a vast body of empirical literature establishing that sentiments, which are completely extrinsic to fundamental factors, can directly influence aggregate outcome both contemporaneously and over a certain time horizon (Benhabib and Spiegel, 2019; Lagerborg et al., 2020; Mian et al., 2015; Levchenko and Pandalai-Nayar, 2020). For example, Benhabib and Spiegel (2019) demonstrate that pure optimistic sentiments can boost real output significantly; and Lagerborg et al. (2020) show that sentiment-driven impacts can persist for a long time. Motivated by these facts, we explore sentiment-driven fluctuations in an otherwise standard real

*We are grateful to Editors Tony He, Pengfei Wang, and Liyan Yang and our discussant Jieran Wu for their valuable comments and suggestions which significantly improve our paper. We also thank conference participants at the 2021 Conference on Markets & Economies with Information Frictions. Zhiwei Xu acknowledges the financial support from the National Natural Science Foundation of China (72022011). Fei Zhou acknowledges the financial support from HKBU One-off Tier 2 Start-up Grant (RC-OFSGT2/20-21/BUS/002). Jing Zhou acknowledges the financial support from Shanghai Institute of International Finance and Economics.

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business cycle (RBC) model. In particular, we consider incomplete information that allows a role of sentiments in agents' decision making, and investigate the joint determination of sentiments and macroeconomic outcomes under rational expectations. We then qualitatively and quantitatively examine the potential power of sentiment shocks in propagating business cycle fluctuations.

In our model, goods markets open after firms' production takes place. When making their production decisions, individual firms receive signals that confound their idiosyncratic demand shocks and market sentiments. Under such an incomplete information structure, individual firms cannot disentangle their fundamentals from market sentiments. A firm's optimal production and investment decisions depend upon the expectation of its idiosyncratic demand and decisions of other firms in the economy. In aggregate, firm-side decisions rely on households' consumption and labor supply decisions, which in turn depend on expected income and market prices that are associated with firms' decisions.

We show that the model has two types of rational expectations equilibria (REE). In a fundamental equilibrium that resembles the saddle path in the standard RBC literature, aggregate outcomes are completely driven by fundamental changes, e.g., technology shocks. In a sentiment-driven equilibrium, the agents' expectations are rational and self-fulfilling regarding the realization of sentiment shocks. As a result, a nonfundamental sentiment can cause fluctuations in the real economy. The sentiment-driven equilibrium hinges on the incomplete information structure on the firms' demand. Optimistic sentiments lead to favorable signals sent to firms. Unable to perfectly disentangle positive idiosyncratic demand shocks from positive sentiments, a firm attributes a favorable signal partially to strong demand for its product and then expands its production and investment. An increase in the total supply of products reduces the aggregate price level and effectively raises real wages and income, stimulating household consumption and labor supply. In the REE, an expansion of aggregate demand on the household side rationalizes the increase of the total supply, resulting in a self-fulfilling sentiment-driven equilibrium. Therefore, in our model, the business cycle fluctuations can purely be driven by waves of pure optimism and pessimism.[1]

In our theoretical analysis, our first contribution is that we demonstrate that the above insight is robust to various modeling details. We first show the existence of the sentiment-driven equilibrium using GHH (Greenwood-Hercowitz-Huffman) preferences. In the absence of the income effect on labor supply, we can obtain tractable solutions and characterize the equilibria in closed forms. We further show that the dynamic paths of aggregate variables in the sentiment-driven REE can be expressed as linear combinations of those in the aggregate

[1] Even though sentiment-driven fluctuations in our paper feature self-fulfilling beliefs about the aggregate outcome, the sentiment equilibria are not simple sunspot randomization over multiple fundamental equilibria. In this sense, our paper is connected with the sunspot equilibrium literature, specifically, those studies showing that sunspot equilibria can occur even when the fundamental equilibrium is unique, (Cass and Shell, 1983; Spear, 1989; Mas-Colell, 1992; Gottardi and Kajii, 1999). However, multiple equilibria in our paper arise from signal extraction problems with endogenous information structures, which largely deviate from the above-mentioned works.

fundamentals and an exogenous sentiment process. We then extend the analysis to a model with a more general form of utility, e.g., KPR (King-Plosser-Rebelo) preferences. We show that the sentiment-driven equilibrium still exists to a first-order approximation, and the equilibrium properties remain valid.

Our second contribution is that we can accommodate our model to generate persistent fluctuations driven by sentiment shocks. When sentiments are persistent over time and firms have information on them in the past, firms can separate the sentiments carried over from the past and only respond to innovations in the sentiments, resulting in short-lived sentiment-driven responses. We show that when information on past periods is contaminated with noises, however, the response of the aggregate economy to a sentiment shock could persist over a certain time horizon, which is consistent with empirical findings in the literature.

We further construct a full-fledged RBC model and quantify the aggregate impact of sentiment shocks. We calibrate the model-specific deep parameters by matching the model-implied moments with those in U.S. aggregate data. The dynamic responses in the calibrated model indicate that sentiment shocks that are orthogonal to the fundamental changes (e.g., technology shocks) boost aggregate fluctuations and drive positive comovements among aggregate output, investment, consumption, and hours worked. We then compare the aggregate volatilities in the sentiment-driven equilibrium with those in the fundamental equilibrium and find that the output volatility in the fundamental equilibrium is 31% smaller than that in the sentiment-driven equilibrium. Moreover, the labor market volatility predicted by the sentiment-driven equilibrium is more empirically reasonable than those predicted by the fundamental equilibrium. These results indicate that sentiment shocks may play an important role in amplyfying real business cycles.

Related literature This paper contributes to the growing literature that analyzes how sentiment shocks transmit to macro-level business cycle fluctuations. First, our paper builds directly on the literature on endogenous sentiments. A partial list includes Benhabib et al. (2015) who look at the production-side incomplete information friction and illustrate how sentiments can generate stochastic self-fulfilling rational expectations equilibrium, Chahrour and Gaballo (2017) that focuses on consumers' incomplete information problem when making their consumption decisions and provides a theory of expectation-driven business cycles in which consumers' learning from prices causes changes in aggregate productivity to shift aggregate beliefs, and Acharya et al. (2021) showing that sentiments alter the volatility and persistence of aggregate outcomes in response to fundamental shocks and provide thorough conditions for this to happen. Benhabib et al. (2015) and Chahrour and Gaballo (2017) study static environments while Acharya et al. (2021) consider a general dynamic framework. Among them, the closest precursor to our paper is Benhabib et al. (2015). However, our paper differs from theirs in that we study a dynamic model in a full-blown business cycle model and explore its qualitative and quantitative potential of accounting for business cycles. Another division of this literature studies the interaction between endogenous sentiments and financial markets. For example, Benhabib et al. (2016) emphasize two-way feedback between the financial sector and the real sector and offer implications for nonlinearity and discontinuity in asset prices. Benhabib et al. (2019) further extend this idea and resolve the paradox introduced by Grossman and Stiglitz (1980).

Our work is also related to the dispersed information literature where sentiments are exogenous shocks to agents' beliefs. Some works along this strand of literature assume sentiments to be common noises in signals that alter agents' first-order beliefs about the fundamentals. For instance, Angeletos and La'O (2010) introduce such information dispersion among firms in an otherwise canonical RBC model and show that technology shocks explain only a small fraction of high-frequency business cycles. Barsky and Sims (2012) study the impulse responses to confidence innovations and "animal spirits shock" which are reflected in the signal commonly received by all agents. Other works in this line of literature assume that sentiment shocks can alter agents' higher-order beliefs about the fundamentals. For example, Angeletos and La'O (2013) study how sentiment shocks can switch higher-order expectations and illustrate the quantitative potential of such shocks in driving business cycles. In addition, Lorenzoni (2009), Acharya (2013), Nimark (2014), Huo and Takayama (2021), Angeletos et al. (2018), Rondina and Walker (2020) among many others, also study expectation-driven fluctuations. To solve these models efficiently and accurately, Han et al. (2019) develop a novel approach for linear rational expectations models and provide an efficient toolbox. In our paper, we focus on sentiments that are endogenously generated and disciplined by rational expectations and establishing sentiment-driven fluctuations in the real economy with analytical solutions.

Finally, our study contributes to the literature on business cycles with sentiments. Angeletos et al. (2018) augment macroeconomic models with higher-order belief dynamics where waves of optimism and pessimism affect the outlook of the economy under incomplete information and frictional coordination. Milani (2017) estimates a dynamic stochastic general equilibrium model with sentiments and finds that sentiments are responsible for a large fraction of business cycle fluctuations. While they deviate from the conventional rational expectation hypothesis, our work stays with rational expectations equilibria.

The remainder of the paper is organized as follows. Section 2 presents the baseline model and defines the rational expectations equilibrium. Section 3 characterizes the fundamental and the sentiment-driven equilibria. Section 4 quantitatively examines the role of sentiment shocks in accounting for real business cycles. Section 5 concludes the paper. Appendices provide more details about the proofs.

2. The Baseline Model

This section describes an otherwise standard RBC model with incomplete information. There are three types of agents in the economy: households, final goods firms, and intermediate goods firms. Households consume and invest final goods and supply labor to the production sector. They own the firms and receive their dividend payments at the end of each period. Final goods firms aggregate intermediate goods into final goods. Intermediate goods firms use capital and labor as inputs to produce differentiated goods. They face idiosyncratic demand shocks and aggregate fundamental shocks, e.g., technology shocks. The key feature of this model is that intermediate goods firms face incomplete information. They make their production and employment decisions before the opening of goods markets and the realization of equilibrium prices.

The timing of events within a period is crucial in our model. We describe it below.

1. At the beginning of each period, the aggregate productivity shock is realized. Households form their sentiments, which are entirely irrelevant to the fundamentals. Based on their expected income and prices, households decide their labor supply and consumption.

2. Observing the aggregate shocks, final goods firms decide their demand for each type of intermediate goods based on their expectation of the prices which will be realized when the goods markets open.

3. Intermediate goods firms observe productivity shocks. They understand that aggregate demand could be driven by both their idiosyncratic demand shocks and sentiment shocks. However, an intermediate goods firm does not directly observe its idiosyncratic demand shock and sentiments. Instead, it receives a signal which is a mixture of these two factors. Based on the received signals, these firms decide their production, or equivalently, their labor inputs for production.

4. The labor market opens. Production of all intermediate goods takes place. We treat labor as a numeraire in the economy and normalize the nominal wage $W_t = 1$.

5. The goods markets open. Intermediate goods are traded, and final goods are aggregated. All market-clearing prices are realized. Households consume and trade firm stocks upon receiving their wages and firms' dividends and profits. Intermediate goods firms make investments. Figure 1 summarizes the timeline of the events.

Given the timing, one period can basically be divided into two subperiods: before the goods markets open (items 1 to 4) and after the goods markets open (item 5). Notice that all agents make their decisions based only on their expectations of the output/income and prices, and there is no guarantee that all the markets clear automatically. However, we will show that in REE, all these markets will always clear.

2.1. Households

Time is discrete and indexed by $t \in \{0, 1, 2, ...\}$. A representative household chooses its consumption flow $\{C_t\}_{t=0}^{\infty}$ of final goods, labor supply $\{N_t\}_{t=0}^{\infty}$ and equity share $\{\psi_{jt}\}_{t=0}^{\infty}$ of an intermediate goods firm j to maximize its life-time expected utility given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, N_t\right),$$

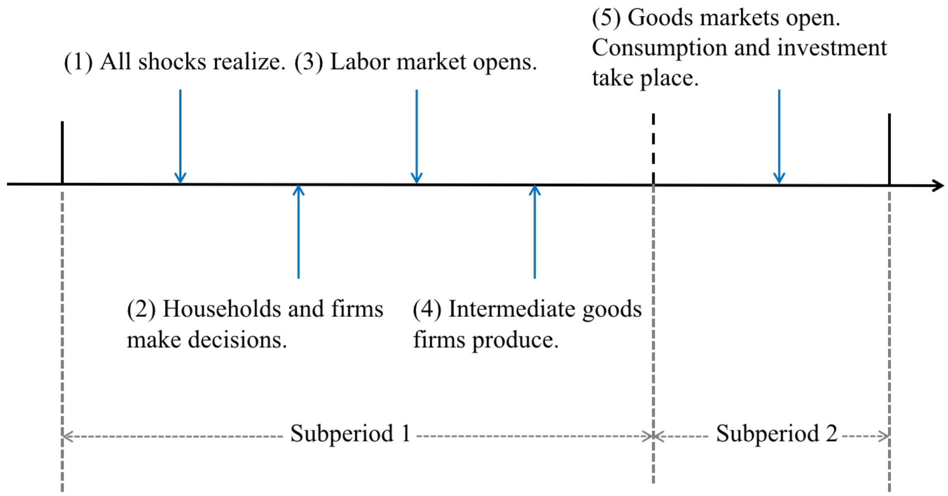


Figure 1: Timeline of Events

subject to the budget constraint

$$C_t + \frac{1}{P_t}\int_0^1 \psi_{jt}(V_{jt} - D_{jt})dj = \frac{1}{P_t}\left(W_t N_t + \Pi_t + \int_0^1 \psi_{jt-1} V_{jt} dj\right).$$

W_t is the nominal wage rate and normalized to 1; P_t is the price level of final goods; V_{jt} is the value of intermediate goods firm j before its dividends D_{jt} are paid and Π_t is total profits from final goods firms. The parameter $\beta \in (0,1)$ denotes the discount factor; and $\delta \in (0,1)$ denotes capital depreciation rate.

Given the timing, households make their decisions about labor supply and consumption before the realization of their income components and the aggregate price level P_t. Let X_t denote the realized value of a variable in equilibrium, i.e., in the second subperiod, and X_t^e the expected value before the realization of X_t, i.e., in the first subperiod. The optimal decisions for labor supply N_t and equity share of an intermediate goods firm j, ψ_{jt}, satisfy

$$0 = U_N\left(C_t^e, N_t\right) + \frac{W_t}{P_t^e} U_C\left(C_t^e, N_t\right), \tag{1}$$

$$V_{jt}^e = D_{jt}^e + \beta \mathbb{E}_t\left[\frac{U_C\left(C_{t+1}, N_{t+1}\right)}{U_C\left(C_t^e, N_t\right)} \frac{P_t^e}{P_{t+1}} V_{jt+1}\right]. \tag{2}$$

A household's labor supply is realized in the first subperiod, whereas the consumption and stock trading occur in the second subperiod. Thus, equations (1) and (2) contain realized labor supply N_t and forecasted values of other variables. When agents' beliefs are rational, then $X_t = X_t^e$ in equilibrium which we already set for future periods.

2.2. Final Goods Firms

The final goods sector is perfectly competitive. A final goods firm uses a continuum of intermediate goods indexed by $j \in [0,1]$ to produce final goods according to a constant elasticity of substitution (CES) aggregation

$$Y_t = \left(\int_0^1 \epsilon_{jt}^{\frac{1}{\theta}} Y_{jt}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}}, \tag{3}$$

where $\theta > 1$ is the elasticity of substitution across different intermediate goods and ϵ_{jt} is an idiosyncratic demand shock for the intermediate goods of type j. We assume that ϵ_{jt} is independent and identically distributed (i.i.d.) and $\varepsilon_{jt} = \log(\epsilon_{jt})$ follows a normal distribution $\mathbf{N}(0, \sigma_\varepsilon^2)$ where $\sigma_\varepsilon > 0$ is the standard deviation. The firm solves the profit optimization problem as follows

$$\max_{Y_t, \{Y_{jt}\}_{j=0}^1} P_t Y_t - \int_0^1 P_{jt} Y_{jt} dj. \tag{4}$$

The optimal demand of final goods firms for each type of intermediate goods Y_{jt} is given by

$$Y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\theta} \epsilon_{jt} Y_t. \tag{5}$$

The demand function increases in firm j's idiosyncratic demand shock ϵ_{jt} and decreases in the relative price of goods P_{jt}/P_t with elasticity θ. Define the price index of final goods P_t as

$$P_t \equiv \left(\int_0^1 \epsilon_{jt} P_{jt}^{1-\theta} dj \right)^{\frac{1}{1-\theta}}. \tag{6}$$

2.3. Intermediate Goods Firms

An intermediate goods firm j operates in a monopolistically competitive market. It uses existing capital stock K_{jt-1} and hires labor N_{jt} to produce intermediate goods according to a Cobb-Douglas production function

$$Y_{jt} = A_t K_{jt-1}^{\alpha} N_{jt}^{1-\alpha}, \tag{7}$$

where $\alpha \in (0,1)$ is the capital share in production and A_t is the aggregate productivity shock. Assume that $a_t = \log(A_t)$ follows an exogenous AR(1) stochastic process $a_t = \rho_a a_{t-1} + \varepsilon_{at}$. Here, $\rho_a \in (-1,1)$ captures the persistence, and $\sigma_a > 0$ is the standard deviation. When the firm decides its labor inputs, the goods market has not yet opened, and all equilibrium goods prices have not realized. Knowing its demand function (5), the intermediate goods firm needs to decide its output based on information available at that moment. However, the firm cannot distinguish the idiosyncratic demand shock ϵ_{jt} from a *sentiment shock* z_t. This sentiment shock reflects households' sentiments on aggregate output and is not necessarily related to fundamentals. In this paper, we assume that the sentiment shocks are independent of productivity shocks and follow a normal distribution, $\mathbf{N}(0,\sigma_z^2)$ where $\sigma_z > 0$ is the standard deviation. The signal s_{jt} received by a firm j is a mixture of the idiosyncratic demand shock ε_{jt} and the sentiment shock z_t,

$$s_{jt} = \lambda\varepsilon_{jt} + (1-\lambda)z_t, \tag{8}$$

where $\lambda \in [0,1]$ is the weight on the demand shock. In other words, the firm j cannot tell a positive demand shock from a positive sentiment shock simply from signal (8). With the normalized wage rate and unrealized equilibrium prices, firms cannot extract information from the input price. Let Ω_{jt} denote the information set faced by the firm j. $\Omega_{jt} = \{s_{jt}\}$ as the shocks ϵ_{jt} and z_t are independent across periods.

We start with a firm's optimal static decisions. Given its predetermined capital stock K_{jt-1}, the intermediate goods firm j solves the following profit maximization problem

$$\Pi_t\left(K_{jt-1}, s_{jt}\right) = \max_{\left\{P_{jt}, Y_{jt}, N_{jt}\right\}} \mathbb{E}\left(P_{jt}Y_{jt} - N_{jt}|s_{jt}\right), \tag{9}$$

subject to its production function (7), demand curve (5) and information structure (8). Though aggregate output has not yet been realized at this moment, the firm believes that aggregate demand will be equal to aggregate output/income, i.e. $Y_t = Y_t^e$, in equilibrium.

With equations (5) and (7), we can replace the labor input N_{jt} and individual price P_{jt} by Y_{jt}. The optimal production of firm j can then be solved as

$$Y_{jt} = \left[\frac{\theta-1}{\theta}(1-\alpha)A_t^{\frac{1}{1-\alpha}}K_{jt-1}^{\frac{\alpha}{1-\alpha}}\mathbb{E}\left(\epsilon_{jt}^{\frac{1}{\theta}}P_tY_t^{\frac{1}{\theta}}|s_{jt}\right)\right]^{\frac{1}{\varrho}}, \tag{10}$$

where $\varrho \equiv \frac{1}{\theta} + \frac{\alpha}{1-\alpha}$. Once goods Y_{jt} are produced, the supply is fixed. When the goods market opens, the demand from final goods firms (5) determines the market-clearing price P_{jt} for this particular type of intermediate goods.

With the asset pricing equation (2), the value of firm j can be expressed as the sum of expected present value of dividend payments

$$V_{jt} = \mathbb{E}_t\sum_{\tau=0}^{\infty}\beta^\tau\frac{U_C\left(C_{t+\tau}, N_{t+\tau}\right)}{U_C\left(C_t, N_t\right)}\frac{P_t}{P_{t+\tau}}D_{jt+\tau}, \tag{11}$$

where the dividends D_{jt} are defined as

$$D_{jt} = \Pi_t\left(K_{jt-1}, s_{jt}\right) - P_t\left[K_{jt} - (1-\delta)K_{jt-1}\right]. \tag{12}$$

Now we turn to the firm's intertemporal decision. The optimal condition for K_{jt} yields the

Euler equation

$$1 = \beta \mathbb{E}_t \left\{ \frac{U_C\left(C_{t+1}, N_{t+1}\right)}{U_C\left(C_t^e, N_t\right)} \left[\frac{1}{P_{t+1}} \frac{\partial \Pi_{t+1}\left(K_{jt}, s_{jt+1}\right)}{\partial K_{jt}} + (1-\delta) \right] \right\}. \tag{13}$$

In a standard RBC model, sentiments which are orthogonal to the fundamentals do not affect firms' investments. In the presence of incomplete information, however, sentiments appear in the signals received by firms and confound their perception of the demand. As we will show later, they can cause aggregate fluctuations in the real economy, which in turn affect the real expected marginal benefits of making investment captured by the right-hand side of the above equation. Given that ε_{jt} in the signal s_{jt} is independent over time, the right-hand side of the above equation only depends on the aggregate conditions, which implies that the desire level of capital stock K_{jt} is identical across all firms, i.e., $K_{jt} = K_t$ for all $j \in [0,1]$.

2.4. Rational Expectations Equilibrium

According to the timing of events, on the one hand, household decisions on consumption and labor supply are based on their expected income and sentiments, while the realized consumption depends on the realized income; on the other hand, the firms' production decisions are based on their expectation about aggregate demand and the price level, while their realized sales revenue depends on all the households' and final goods firms' actions. At the moment that intermediate goods firms make production decisions, the goods market has not yet opened, and there is no guarantee that the demand of final goods will automatically meet the supply.

In a REE, however, for any joint realization of (a_t, z_t), all aggregate quantities and prices in equilibrium turn out to coincide with their values under rational expectations, i.e., $X_t = X_t^e$ for all endogenous variables. The definition for REE is given below.

Definition 1. *A REE is a sequence of allocations* $\{C_t, N_t, K_t, Y_t, \Pi_t, \{Y_{jt}\}_{j\in[0,1]}, \{N_{jt}\}_{j\in[0,1]}, \{K_{jt}\}_{j\in[0,1]}, \{D_{jt}\}_{j\in[0,1]}$, *prices* $\{P_t, \{P_{jt}\}_{j\in[0,1]}, W_t = 1\}$, *and a distribution of* z_t, $\mathbb{F}(z_t)$, *such that for each joint realization of* (a_t, z_t),

(i) a household maximizes its utility given the equilibrium prices $W_t = 1$ *and* P_t, *profits* Π_t, *dividends* $\{D_{jt}\}_{j\in[0,1]}$ *and stock prices* $\{V_{jt}\}_{j\in[0,1]}$;

(ii) a final goods firm maximizes its profits given the equilibrium prices P_t *and* $\{P_{jt}\}_{j\in[0,1]}$;

(iii) an intermediate goods firm maximizes its expected profits given the equilibrium prices P_t, P_{jt}, $W_t = 1$ *and the signal in (8);*

(iv) all markets clear, i.e., $N_t = \int_0^1 N_{jt} dj$ *for the labor market and*

$$C_t + K_t = Y_t + (1-\delta) K_{t-1}, \tag{14}$$

for the final goods market;

(v) beliefs are rational such that $P_t^e = P_t$, $\Pi_t^e = \Pi_t$, $C_t^e = C_t$, and $Y_t^e = Y_t$.

3. Equilibrium Characterization

In this section, we characterize two types of equilibria in our model: fundamental equilibrium and sentiment-driven equilibrium. We will show that, in the fundamental equilibrium, macroeconomic fluctuations are only driven by aggregate productivity shocks, whereas in a sentiment-driven equilibrium, sentiment shocks that are orthogonal to the fundamentals can also generate macroeconomic fluctuations as productivity shocks.

3.1. Equilibria under GHH Preferences

For illustrative purpose, we use the GHH utility function which removes the wealth effect on labor supply and simplifies the algebra. In particular, the utility function takes the form of $U(C_t, N_t) = \log\left(C_t - \varphi \frac{N_t^{1+\nu}}{1+\nu}\right)$, where $\varphi > 0$ is the weight on disutility of labor supply and $\nu \geq 0$ is the inverse of Frisch elasticity.

Under GHH utility, the labor supply condition (1) becomes

$$\varphi N_t^{\nu} = \frac{W_t}{P_t^e}. \tag{15}$$

With normalization $W_t = 1$ and rational expectations, the above condition says that movements in aggregate labor supply are one-to-one mapped to movements in the aggregate price. When the inverse of Frisch elasticity is zero, i.e., $\nu = 0$, the aggregate price becomes a constant. Thereby, the aggregate price fluctuates only when $\nu > 0$.

3.1.1. Fundamental Equilibrium

First, we consider the REE under perfect information. In this equilibrium, firms can perfectly observe the aggregate productivity shock A_t, the sentiment shock z_t and the idiosyncratic demand shock ϵ_{jt}. Intermediate goods firms do not need to extract information from the signal, i.e., $\mathbb{E}\left(\epsilon_{jt}^{\frac{1}{\theta}} P_t Y_t^{\frac{1}{\theta}} | s_{jt}\right) = \epsilon_{jt}^{\frac{1}{\theta}} P_t Y_t^{\frac{1}{\theta}}$. The optimal production decision (10) becomes

$$Y_{jt} = \kappa_y \left(A_t^{\frac{1}{1-\alpha}} K_{jt-1}^{\frac{\alpha}{1-\alpha}} \epsilon_{jt}^{\frac{1}{\theta}} P_t Y_t^{\frac{1}{\theta}} \right)^{\frac{1}{\varrho}}, \tag{16}$$

where $\kappa_y = \left[\left(\frac{\theta-1}{\theta}\right)(1-\alpha)\right]^{\frac{1}{\varrho}}$.

From the production function (7) and labor market clearing condition, the optimal labor demand can be written as

$$N_{jt} = \frac{\exp\left[\frac{1}{1+\alpha(\theta-1)}\varepsilon_{jt}\right]}{\int_0^1 \exp\left[\frac{1}{1+\alpha(\theta-1)}\varepsilon_{jt}\right] dj} N_t, \tag{17}$$

where the aggregate labor satisfies

$$N_t = \mu \kappa_y^{\frac{1}{1-\alpha}} A_t^{\frac{\alpha(\theta-1)}{1+\alpha(\theta-1)}} K_{t-1}^{\frac{\alpha(\theta-1)}{1+\alpha(\theta-1)}} \left(P_t Y_t^{\frac{1}{\theta}}\right)^{\frac{\theta}{1+\alpha(\theta-1)}}, \tag{18}$$

where $\mu = \int_0^1 \exp\left[\frac{1}{1+\alpha(\theta-1)}\varepsilon_{jt}\right] dj$ is the mean of the idiosyncratic shock $\epsilon_{jt}^{\frac{1}{1+\alpha(\theta-1)}}$. From the CES aggregation function (3), we can derive aggregate output as

$$Y_t = \mu^{\frac{1+\alpha(\theta-1)}{\theta-1}} A_t K_{t-1}^{\alpha} N_t^{1-\alpha}. \tag{19}$$

Given the log-normal distribution of ε_{jt}, we have $\mu^{\frac{1+(\theta-1)\alpha}{\theta-1}} = \exp\left[\frac{1}{(1-\alpha+\alpha\theta)(\theta-1)}\frac{\sigma_\varepsilon^2}{2}\right]$. To facilitate the presentation, we use lower case to label the logarithm of variables, i.e., $x_t = \log(X_t)$, superscript **f** to label variables in the fundamental equilibrium and superscript **s** to label variables in the sentiment-driven equilibrium.

Denote the policy function of aggregate output as $\mathbf{G}_t\left(a_t, k_{t-1}\right)$. From the labor supply curve (15), aggregate labor demand (18), aggregate output (19), and $W_t = 1$, we can solve $\mathbf{G}_t$ in the fundamental equilibrium as

$$y_t^{\mathbf{f}} = \mathbf{G}_t^{\mathbf{f}}\left(a_t, k_{t-1}\right) = \Xi_y^{\mathbf{f}}\left(\mu, \alpha, v, \theta, \varphi\right) + \mathbf{\Lambda}_y^{\mathbf{f}}(\alpha, v)\left(a_t + \alpha k_{t-1}\right), \tag{20}$$

where $\Xi_y^{\mathbf{f}}$ is a constant depending on μ and other structural parameters $\{\alpha, v, \theta, \varphi\}$; and the coefficient $\mathbf{\Lambda}_y^{\mathbf{f}}(\alpha, v) = \frac{1+v}{\alpha+v}$. Given $\mathbf{G}_t^{\mathbf{f}}$, aggregate labor n_t and the price p_t can be solved from (15) and (18) jointly. Finally, the optimal capital $k_{jt} = k_t$ is determined by the Euler equation (13). Appendix A.1 provides the detailed derivations of this equilibrium.

The policy function (20) indicates that aggregate output is proportional to the fundamental component $a_t + \alpha k_{t-1}$, implying that output perfectly reveals the fundamentals. We call this equilibrium the *fully revealing REE*. The following Proposition 1 describes the fundamental equilibrium.

Proposition 1. *There exists a unique fundamental equilibrium in which the aggregate endogenous variables $\{p_t, y_t, n_t, k_t, c_t\}$ are characterized by the equation system (13), (14), (15), (18) and (20). The usual transversality condition holds.*

Proof. See Appendix A.1 for details. ■

Given the policy function $\mathbf{G}_t^{\mathbf{f}}$, an individual firm's decisions on $\{y_{jt}, n_{jt}, p_{jt}\}$ are uniquely pinned down by (5), (16) and (17), respectively.

The above model nests two special cases. When the capital share $\alpha = 0$, the model degenerates to the setup in Benhabib et al. (2015), which is free of capital and essentially a static environment. In the case of indivisible labor, i.e., $\nu = 0$, the price level becomes a constant,[2] and y_t and n_t are proportional to k_{t-1}.

3.1.2. Sentiment-Driven Equilibria

We now explore the equilibria in which intermediate goods firms have incomplete information. In this case, the firms cannot precisely disentangle idiosyncratic fundamental shocks ε_{jt} from sentiment shocks z_t in the noisy signals. They attribute a fraction of their observed signals to their idiosyncratic demand, regardless of whether they are caused by demand shocks or sentiment shocks. They make their production decisions responding to these signals. Sentiments can affect their decisions collectively and hence aggregate output. By understanding that, firms rationally believe that sentiments can drive fluctuations in aggregate demand. As a result, there exist sentiment-driven equilibria that are different from the fundamental equilibrium. The following proposition characterizes a sentiment-driven equilibrium of this model.

Proposition 2. *Assume that sentiment shocks z_t and fundamental shocks a_t and ε_{jt} are independent of each other and over time. Let $\lambda \in (0, \frac{1}{2})$ and $\frac{1-2\lambda}{\lambda} > \frac{\alpha}{1-\alpha}\theta$. There exists a sentiment-driven equilibrium, in which the policy function of output, $\mathbf{G}_t^{\mathbf{s}}(a_t, k_{t-1}, z_t)$, satisfies*

$$y_t^{\mathbf{s}} = \mathbf{G}_t^{\mathbf{s}}(a_t, k_{t-1}, z_t) = \boldsymbol{constant} + \mathbf{G}_t^{\mathbf{f}}(a_t, k_{t-1}) + z_t, \tag{21}$$

where $\mathbf{G}_t^{\mathbf{f}}(a_t, k_{t-1})$ is the policy function in the fundamental equilibrium given by (20); and σ_z satisfies $\frac{1-\lambda}{\varrho}\frac{\frac{\lambda}{\theta}\sigma_\varepsilon^2 + \left(\frac{1}{\theta} - \frac{\nu}{1-\alpha}\right)(1-\lambda)\sigma_z^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2} = 1$.

Given any joint realization of (a_t, z_t), the endogenous aggregate variables $\{c_t, n_t, y_t, k_t, p_t\}$ are characterized by the equation system (3), (13), (14), (15) and (21).

Proof. See Appendix A.2 for details. ∎

Conceptually, to derive the policy function $\mathbf{G}_t^{\mathbf{s}}(a_t, k_{t-1}, z_t)$ is essentially to solve a fixed point problem. Intermediate goods firms form their own beliefs on the aggregate output dynamics, which are linear in the fundamentals $\{a_t, k_{t-1}\}$ and sentiment shocks z_t. To determine its optimal production, an individual firm needs to infer a compounded term consisting of its idiosyncratic demand ϵ_{jt} and the aggregate conditions P_t and Y_t, $\mathbb{E}\left(\epsilon_{jt}^{\frac{1}{\theta}} P_t Y_t^{\frac{1}{\theta}} | s_{jt}\right)$. With the information structure, this term can be expressed as the product of an observed fundamental component including $\{a_t, k_{t-1}\}$ and the expectation of an unobserved component

[2] Note that the constant price level is associated with GHH preferences. Under KPR preferences, the price level still varies with indivisible labor.

$\exp[x_{jt}(\varepsilon_{jt}, z_t)]$, where $x_{jt}(\varepsilon_{jt}, z_t)$ is a linear function of ε_{jt} and z_t. With firms' forecast on the dynamics of aggregate output and the price level, the individual firm j infers the unobserved component $x_{jt}(\varepsilon_{jt}, z_t)$ based on the signal s_{jt} by solving a signal extraction problem. Then the firm j decides its optimal employment, production and investment. Aggregating the actions of all individuals gives the realized dynamics of aggregate output and the price level which should be consistent with the initial forecast, forming a sentiment-driven REE.

To be more specific, in Appendix A.2.1 we show that in the sentiment-driven equilibrium, an individual firm j's optimal production y_{jt} can be written as the following best response function:

$$y_{jt} = \frac{1}{\varrho}\frac{1}{1-\alpha}\left(a_t + \alpha k_{jt-1}\right) + \frac{1}{\varrho}\left(\frac{1}{\theta} - \frac{v}{1+v}\right)\mathbf{\Lambda}_y^{\mathbf{f}}(\alpha, v)\left(a_t + \alpha k_{t-1}\right) + \frac{1}{\varrho}\Upsilon_{\mathbf{x}} s_{jt} + \text{constant}, \quad (22)$$

where $\Upsilon_{\mathbf{x}} = \frac{\frac{\lambda}{\theta}\sigma_\varepsilon^2 + \left(\frac{1}{\theta} - \frac{v}{1-\alpha}\right)(1-\lambda)\sigma_z^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2}$ is the signal-noise ratio obtained from the signal extraction problem $\mathbb{E}\left(x_{jt}|s_{jt}\right)$. The first term in the right-hand side of the above equation reflects the response of its optimal decision to its own fundamental $a_t + \alpha k_{jt-1}$. The second term comes from the firm's forecast on the dynamics of aggregate price p_t and output y_t in the REE. The forecast rule of output takes the form given by (21), as the forecasted output must be equal to the realized value in the REE. We can show that, as (21), the policy functions of output and aggregate price are simply linear functions of the aggregate fundamental $a_t + \alpha k_{t-1}$ and the sentiment shock z_t. Since a_t and k_{t-1} are observable to firms, the responses to these two components can be singled out while the response to the sentiment shock z_t is embedded in the third term. The third term captures the information extraction and the firm j's best response to the signal s_{jt}.

For the REE to hold, goods and labor market must clear for any joint realization of (a_t, z_t), which requires that the standard deviation σ_z should satisfy $\frac{1-\lambda}{\varrho}\Upsilon_{\mathbf{x}} = 1$, where we normalize the magnitude of a sentiment-driven movement in output y_t to be the same as that of the movement in sentiments z_t, as indicated by (21). This condition determines σ_z endogenously. More generally, there is a one-to-one correspondence between the magnitude of output response to sentiments and the standard deviation σ_z, implying the existence of an infinite number of sentiment-driven REEs which are similar to the one characterized by this proposition. If the parameters imply $\sigma_z^2 < 0$, then the sentiment-driven equilibrium does not exist.

Proposition 2 characterizes a REE in which sentiment shocks that are orthogonal to fundamental shocks can also drive aggregate fluctuations. The intuition is as follows. Optimistic sentiments from households cause positive signals received by the firms. With their forecast rules on aggregate demand and the price level, firms attribute a fraction of the favorable signals to increases in their idiosyncratic demand shocks and the rest to aggregate sentiments. The sentimental component coordinates these firms' best responses and thus raises the production of all types of intermediate goods, as captured by the term $\frac{1}{\varrho}\Upsilon_{\mathbf{x}} s_{jt}$ in equation (22). This positive impact of sentiments on the supply of intermediate goods reduces their prices and the aggregate price level. A lower price level stimulates the demand for final goods, which meets the increase in aggregate supply through the market clearing condition. As a result, despite being orthogonal to fundamental shocks, sentiment shocks are also rationalized and can drive business cycles.

3.1.3. Stability Under Learning

We now examine whether the fundamental and sentiment-driven equilibria are stable under learning. To construct the equilibrium in the learning dynamics, we follow Benhabib et al. (2015) and assume that firms perceive the process of aggregate output as

$$y_t^{\mathbf{l}} = \bar{y} + \mathbf{\Lambda}^{\mathbf{l}} \left(a_t + \alpha k_{t-1} \right) + \sigma_{zt} z_t^{\mathbf{l}}, \tag{23}$$

where $\bar{y}$ is a constant consisting of parameters, $z_t^{\mathbf{l}}$ is a standard normal random variable, and σ_{zt} is the firms' perceived value of the standard deviation of sentiment shocks. Define $\tilde{y}_t \equiv y_t^{\mathbf{l}} - \bar{y} - \mathbf{\Lambda}^{\mathbf{l}} \left(a_t + \alpha k_{t-1} \right)$. In this learning model, firms do not know the exact value of the standard deviation σ_z. However, they understand that $\tilde{y}_t$ is proportional to the sentiment shock $z_t^{\mathbf{l}}$ in equilibrium so that they could learn σ_z by iteratively learning σ_{zt}.

By solving the REE (Appendix A.2.2), we can show that under the forecast rule (23), $\tilde{y}_t$ must satisfy

$$\tilde{y}_t = \frac{1-\lambda}{\varrho} \Upsilon_{\mathbf{x}} \sigma_{zt} z_t^{\mathbf{l}}. \tag{24}$$

Following Evans and Honkapohja (2012), the firms update σ_{zt} with the following rule,

$$\sigma_{zt+1} = \left(1 - g \right) \sigma_{zt} + g \frac{\tilde{y}_t}{z_t^{\mathbf{l}}}, \tag{25}$$

where $g \in (0, 1)$ is a constant gain. The above two equations determine the dynamics of σ_{zt} as

$$\sigma_{zt+1} \equiv h(\sigma_{zt}) = \left(1 - g \right) \sigma_{zt} + g \frac{1-\lambda}{\varrho} \Upsilon_{\mathbf{x}} \sigma_{zt}. \tag{26}$$

Echoing the discussion in the previous two sections, there exist two solutions of σ_z^* that solve the fixed point problem $\sigma_z^* = h(\sigma_z^*)$. The first solution is $\sigma_z^* = 0$, corresponding to the fundamental equilibrium. The second solution satisfies $\sigma_z^* > 0$, corresponding to the sentiment-driven equilibrium. More importantly, in Appendix A.3 we verify that the sentiment-driven equilibrium is stable under learning when the learning gain g is sufficiently small.[3] This result validates our focus on the sentiment-driven equilibrium and is in accordance with the stability studied in Benhabib et al. (2015) and Acharya et al. (2021).

[3] According to the E-stability principle (Evans and Honkapohja, 2012), the equilibrium is stable if and only if $|h'(\sigma_z^*)| < 1$. Appendix A.3 also shows that the fundamental equilibrium is not stable under learning since $|h'(0)| > 1$.

3.2. Equilibria under General Preferences

We now extend our analysis to a model with a more general form of preferences. In particular, we consider preferences with an increasing and concave utility function of $U(C_t, N_t)$ that satisfies standard regularity conditions. The full dynamic system for $\{C_t, N_t, K_t, Y_t, P_t\}$ is similar to that in the case of GHH preferences except that the labor supply curve becomes

$$\frac{U_C\left(C_t^e, N_t\right)}{P_t} = \varphi N_t^{\nu}. \tag{27}$$

Since there is no analytical representation of the policy functions with a general utility function, we solve the two types of equilibria based on their log-linearized systems around their corresponding steady states.

Fundamental Equilibrium The fundamental equilibrium system is summarized by (13), (14), (18), (19) and (27). Let the vector of control variables $\hat{\mathbf{X}}_t = [\hat{p}_t, \hat{y}_t, \hat{n}_t, \hat{c}_t]'$ and the vector of state variables $\hat{\mathbf{S}}_t = \left[a_t, \hat{k}_{t-1}\right]'$, where $\hat{x}_t = \log(X_t) - \log(X^{\mathbf{f}})$ is the percentage deviation of a variable X_t from its fundamental steady-state value $X^{\mathbf{f}}$. Appendix B.1 provides the derivation for a linearized version of the fundamental equilibrium system, which can be expressed as

$$\mathbb{A}^{\mathbf{f}} \begin{bmatrix} \hat{\mathbf{S}}_t^{\mathbf{f}} \\ \hat{\mathbf{X}}_t^{\mathbf{f}} \end{bmatrix} = \mathbb{B}^{\mathbf{f}} \mathbb{E}_t \begin{bmatrix} \hat{\mathbf{S}}_{t+1}^{\mathbf{f}} \\ \hat{\mathbf{X}}_{t+1}^{\mathbf{f}} \end{bmatrix}, \tag{28}$$

where $\mathbb{A}^{\mathbf{f}}$ and $\mathbb{B}^{\mathbf{f}}$ are coefficient matrices of this log-linearized system that depend on the deep parameters and the fundamental steady state. This dynamic system essentially characterizes a standard RBC model. Therefore, there exists a unique REE, where the policy function of the aggregate endogenous variables take the form of

$$\begin{bmatrix} \hat{\mathbf{X}}_t^{\mathbf{f}} \\ \hat{k}_t^{\mathbf{f}} \end{bmatrix} = \mathbf{\Lambda}^{\mathbf{f}} \hat{\mathbf{S}}_t^{\mathbf{f}}, \tag{29}$$

where $\mathbf{\Lambda}^{\mathbf{f}}$ is the coefficient matrix obtained from the standard procedure of solving a RBC model.

Sentiment-Driven Equilibrium When solving the sentiment-driven equilibrium, we employ a similar guess-and-verify approach for the forecast rules of the macro aggregate variables similar to that in the case of GHH preferences . In particular, intermediate goods firms conjecture that the process of aggregate control variables $\hat{\mathbf{X}}_t^{\mathbf{s}} = [\hat{p}_t, \hat{y}_t, \hat{n}_t, \hat{c}_t]'$ jointly follow

$$\begin{bmatrix} \hat{\mathbf{X}}_t^{\mathbf{s}} \\ \hat{k}_t^{\mathbf{s}} \end{bmatrix} = \mathbf{\Lambda}^{\mathbf{s}} \hat{\mathbf{S}}_t^{\mathbf{s}} + \mathbf{\Theta}^{\mathbf{s}} z_t, \tag{30}$$

where $\mathbf{\Lambda^s}$ and $\mathbf{\Theta^s}$ are coefficient matrices to be determined.

With this forecast rule, an individual firm's labor and production decisions $\{n_{jt}, y_{jt}\}$ can be expressed as linear functions of its own state variables, $\left[a_t, \hat{k}_{jt-1}\right]'$, and its forecast for the aggregate economy conditional on the signal it receives, $\mathbb{E}\left(\hat{\mathbf{X}}_t^{\mathbf{s}} | s_{jt}\right)$. The aggregation of individual decisions of $\{n_{jt}, y_{jt}\}$, Euler equation of investment decision (13), resource constraint (14) and labor supply condition (27) constitute a dynamic system that determines the policy function

$$\left[\begin{array}{c} \hat{\mathbf{X}}_t^{\mathbf{s}} \\ \hat{k}_t^{\mathbf{s}} \end{array}\right] = \mathbf{G}\left(\mathbf{\Lambda^s}, \mathbf{\Theta^s}, \mathbf{X^s}\right)\left[\begin{array}{c} \hat{\mathbf{S}}_t^{\mathbf{s}} \\ z_t \end{array}\right]. \tag{31}$$

where the vector $\mathbf{X^s} = [P^{\mathbf{s}}, Y^{\mathbf{s}}, N^{\mathbf{s}}, C^{\mathbf{s}}, K^{\mathbf{s}}]'$ collects steady-state values of the aggregate variables. Matching the coefficients in the conjecture rule (30) and the policy function (31) yields restrictions on the elements in matrices $\mathbf{\Lambda^s}$ and $\mathbf{\Theta^s}$. Combining these conditions with the steady-state conditions can uniquely pin down $\mathbf{\Lambda^s}$, $\mathbf{\Theta^s}$ and $\mathbf{X^s}$. This procedure solves the sentiment-driven REE. We relegate the derivation details to Appendix B.2.

It is worth noting that when the volatility of sentiment shocks $\sigma_z \to 0$ and a signal s_{jt} precisely reflects idiosyncratic demand ε_{jt} (i.e., $\lambda \to 1$), the above sentiment-driven REE converges to the fundamental equilibrium described by (29). To see this, under the forecast rule (30), we can write the linearized sentiment-driven equilibrium system as

$$\mathbb{A}^{\mathbf{s}}\left[\begin{array}{c} \hat{\mathbf{S}}_t^{\mathbf{s}} \\ \hat{\mathbf{X}}_t^{\mathbf{s}} \end{array}\right] = \mathbb{B}^{\mathbf{s}}\mathbb{E}_t\left[\begin{array}{c} \hat{\mathbf{S}}_{t+1}^{\mathbf{s}} \\ \hat{\mathbf{X}}_{t+1}^{\mathbf{s}} \end{array}\right] + \mathbb{C}^{\mathbf{s}} z_t. \tag{32}$$

where $\mathbb{A}^{\mathbf{s}}$ and $\mathbb{B}^{\mathbf{s}}$ are matrices analogous to $\mathbb{A}^{\mathbf{f}}$ and $\mathbb{B}^{\mathbf{f}}$ in (28), and vector $\mathbb{C}^{\mathbf{s}}$ collects all the coefficients of $\hat{\mathbf{S}}_t^{\mathbf{s}}$ and $\hat{\mathbf{X}}_t^{\mathbf{s}}$ before z_t. By comparing the steady-state conditions of the fundamental and sentiment-driven equilibria, it is straightforward to verify that when $\sigma_z \to 0$ and $\lambda \to 1$, the steady-state values of the aggregate variables in the sentiment-driven equilibrium converge to their counterparts in the fundamental one. Consequently, we have $\mathbb{A}^{\mathbf{s}} \to \mathbb{A}^{\mathbf{f}}$ and $\mathbb{B}^{\mathbf{s}} \to \mathbb{B}^{\mathbf{f}}$ when $\sigma_z \to 0$ and $\lambda \to 1$. We then have the following proposition.

Proposition 3. *Assume that sentiment shocks z_t and fundamental shocks a_t and ε_{jt} are independent of each other and over time. Under more general preferences, there could exist a sentiment-driven REE satisfying (30) and σ_z is endogenously determined. Moreover, the policy function of aggregate variables in the sentiment-driven REE is a linear combination of that in the fundamental equilibrium and the sentiments, when the sentiment shocks are small and signals are precise.*

Proof. See Appendix B.2 for details. ■

This proposition indicates that, whenever the fundamental equilibrium has a saddle path around its steady state, the sentiment-driven equilibrium also has a saddle path around its steady state if the sentiment volatility is small. As a result, the policy function of $\hat{\mathbf{X}}_t^{\mathbf{s}}$ can be written as a linear combination of the state variables a_t and $\hat{k}_{t-1}$ and the sentiment term z_t as that in (30). When $\sigma_z \to 0$ and $\lambda \to 1$, we can further approximate $\mathbf{\Lambda^s}$ in (30) by $\mathbf{\Lambda^f}$. This approximation can largely facilitate the solution procedure of sentiment-driven equilibrium in a quantitative analysis. Meanwhile, the standard deviation σ_z is endogenously determined by the requirements that all the markets must clear in the REE for any joint realization of (a_t, z_t), if the parameters allow $\sigma_z > 0$. Note that the above insight does not rely on specific types of preferences, corroborating the robust existence of the sentiment-driven equilibrium with a large variety of utility functions.

3.3. Persistence of Sentiment-Driven Fluctuations

In the previous analysis, we assume that sentiment shocks are i.i.d. across periods. In this dynamic setting, the impact from a one-time shock could last for more than one period via households' consumption-saving decisions. Nonetheless, since a sentiment shock only confounds the signal in the current period, such impact arises as a general equilibrium effect and diminish rapidly across periods , which deviates from the empirical findings in the literature (e.g., Lagerborg et al., 2020).

In this section, we consider the case where sentiment shocks have time persistence. Without loss of generality, we assume that z_t follows an AR(1) process $z_t = \rho_z z_{t-1} + \varepsilon_{zt}$ where $\rho_z \in (-1, 1)$ captures the degree of persistence. The persistent impact of sentiment shocks requires an incomplete information structure on the history of z_t realization. This is because if having the information on the history $\{z_{t-\tau}\}_{\tau=1}^{\infty}$ or essentially z_{t-1}, the firm can easily abstract z_{t-1} away from its signal s_{jt} when solving the signal extraction problem. After separating the component z_{t-1} and knowing that it has nothing to do with the fundamentals, firms do not respond to z_{t-1} at all, just as in the fundamental equilibrium. Hence, only the innovation term in the sentiment shock, ε_{zt}, plays a role when firms infer the aggregate economic condition from the signals, which leads to similar aggregate impacts of sentiment shocks where they are i.i.d.

We now introduce incomplete information to the history of the sentiment process by assuming that firms observe past realizations of sentiments with noises, as suggested in Acharya et al. (2021). This assumption can be motivated by empirical evidence that agents do not have precise information on each past period (Coibion and Gorodnichenko, 2015). Specifically, the firms cannot precisely observe the history of sentiments up to L periods in the past and have accurate information on sentiments before period $t - L$. Therefore, the information set Ω_{jt} of firm j becomes

$$\mathbf{\Omega}_{jt} = \left\{ \lambda \varepsilon_{jt} + (1-\lambda) z_t, \mathbf{Z}_{t-1}^{L} + \mathbf{v}_{jt-1}^{L}, z_{t-L-1} \right\}, \tag{33}$$

where L is a given positive integer; $\mathbf{Z}_{t-1}^{L} = \left[z_{t-1}, z_{t-2}, ..., z_{t-L}\right]'$; $\mathbf{v}_{jt-1}^{L} = \left[v_{jt-1}, v_{jt-2}, ..., v_{jt-L}\right]'$ and $\{v_{jt-\tau}\}_{\tau=1}^{L}$ are noise terms following normal distributions.

Using a similar solution procedure described in Section 3.2 and replacing the information set by (33), we can show that in the sentiment-driven REE, aggregate variables not only depend on the contemporaneous innovation in sentiments ε_{zt}, but also on past innovations $\{\varepsilon_{zt-\tau}\}_{\tau=1}^{L}$, resulting in a persistent impact of sentiments on aggregate dynamics. Understanding this, firms need to take sentiments into account, when deriving their marginal real profits in the future, as captured by the term $\partial\Pi_{t+1}(K_{jt}, s_{jt+1})/\partial K_{jt}/P_{t+1}$ in equation (13). Intuitively, given other things equal, persistent optimistic sentiments imply stronger aggregate demand and thus larger profits in the future, which encourage firms' investments. As a consequence, this setup strengthens the impacts of sentiments on capital accumulation which is translated to business cycle fluctuations. We quantitatively document the above property in the next section and relegate the detailed derivation to Appendix C.

Learning from past endogenous variables In this setup, we do not allow firms to learn from the past realization of aggregate endogenous variables, which can be used by firms to accurately infer the past realization of sentiments $\{z_{t-\tau}\}_{\tau=1}^{L}$.[4] However, in Appendix D we show that the information structure that allows the firms to have noisy information on the history of endogenous variables rather than sentiment shocks is isomorphic to (33). The underlying rationale is that firms can utilize the policy function described by (31) and translate the noisy information on the history of endogenous variables to noisy information on the history of exogenous variables (sentiments), which is essentially the case in (33). By the same token, as long as the information is noisy, the sentiment-driven fluctuations are still persistent even when the firms could learn the history of both exogenous and endogenous variables. Since our main focus is how sentiments alter business cycle fluctuations rather than how information on the history is aggregated via endogenous market variables, we proceed with the exogenous information structure presented in (33).[5]

4. Quantitative Analysis

After establishing the robust existence of sentiment-driven equilibria in a real business cycle model, we next ask: how quantitatively important are the sentiment shocks in amplifying business cycles? We address this question through a quantitative analysis based on a full-fledged model with the information structure given by (33). We adopt a KPR utility $U(C_t, N_t) = \log(C_t) - \varphi\frac{N_t^{1+\nu}}{1+\nu}$ and solve the model with the procedure described in Section 3.2. We then parameterize the deep parameters through a standard calibration procedure. Based on the calibrated model, we compute the dynamic responses of aggregate variables and business cycle moments.

[4] In principle, the information structure in this model can be extended to include signals containing endogenous variables. However, generic closed-form solutions are not available, and one may need to resort to numerical solutions to solve the model. Han et al. (2019) provide an efficient and accurate method and associated toolbox for solving this type of models.

[5] See Huo and Takayama (2021) for more discussion on why endogenous information equilibrium can be viewed as particular exogenous information equilibrium

4.1. Calibration

We calibrate the model to the U.S. economy. One period in the model corresponds to a quarter. We divide the parameters into two subsets. The first subset of parameters, $\{\beta, \alpha, \nu, \delta, \theta, \varphi\}$, are calibrated to stylized facts or set at standard values from the literature. We follow the RBC literature to set the discount factor $\beta = 0.99$, the capital share in production $\alpha = 0.3$, the depreciation rate of capital stock $\delta = 0.025$. We follow Angeletos et al. (2020) and others to set the substitution elasticity of intermediate goods, θ, to be 7.5 such that the markup is 15%. We set ν to 1.87 such that Frisch elasticity of labor supply ν is 0.53, which lies within the estimated range in the literature. The value of the weight on labor disutility, φ, is chosen such that the steady-state hours worked is 0.25, implying $\varphi = 39.80$.

To generate smooth responses to sentiment shocks in the quantitative analysis, we assume that the noises on the past realization of sentiments decay as time goes backward. That is, the earlier a period is, the more precise the related information is. For simplicity, we assume that $\{v_{jt-\tau}\}_{\tau=1}^{L}$ are i.i.d. and follow normal distributions $\mathbf{N}(0, \sigma_{v,\tau}^2)$ where $\sigma_{v,\tau} = \gamma^{\tau-1}\sigma_v$. The parameter $\gamma \in (0,1)$ measures the decay rate of noises. Notice that the choice of L does not matter. This is because when going back to a very early period, as noises almost fade away, the related information could be fairly precise, where the computation on the history is effectively truncated. In our quantitative analysis, we set $L = 20$ and have verified that our result is insensitive to the value of L.

The second subset of parameters, $\{\rho_a, \sigma_a, \rho_z, \sigma_z, \sigma_\varepsilon, \sigma_v, \lambda, \gamma\}$, includes the parameters of the persistence and standard deviation in the AR(1) processes for the technology shock a_t and sentiment shock z_t, $\{\rho_a, \sigma_a, \rho_z, \sigma_z\}$, the standard deviations of the idiosyncratic demand ε_{jt} and noises v_{jt}, $\{\sigma_\varepsilon, \sigma_v\}$, the weight of ε_{jt} in the signal, λ, and the decay rate of noises, γ. We jointly calibrate these eight parameters by minimizing the distance between the moments generated from the REE with sentiments and those in the real data. The calibration procedure regarding the second subset of parameters proceeds as follows.

First, we do not impose that the magnitude of the sentiment-driven movement in output has to be the same as that of the movement in sentiments as in Proposition 3. Instead, we vary the value of σ_z trying to reproduce the fact that a one-standard-deviation increase in sentiments boosts output by approximately 1.1%, as documented in Benhabib and Spiegel (2019).[6] We then solve all of the coefficients in the policy function (30) subject to the discipline of rational expectations.

Second, we employ the correlation between the actual and the forecasted value of three-quarter-ahead output and investment, which are $\text{corr}(\hat{y}_{t+3}, \mathbb{E}_t\hat{y}_{t+3})$ and $\text{corr}(\hat{i}_{t+3}, \mathbb{E}_t\hat{i}_{t+3})$ in the model. These two moments reflect the persistence of productivity and sentiment shocks under rational expectations. The forecast data are forecasts of the quarterly chained-weighted

[6]In Benhabib and Spiegel (2019), this estimate ranges from 1.1% to 3.6%. The authors think the lower end is more plausible.

real GDP (RGDP) and nonresidential investment (RNRESIN) from the Survey of Professional Forecasters (SPF) by the Federal Reserve Bank of Philadelphia, which reports forecasts for outcomes in the current and next four quarters, typically about the level of the variable in each quarter.

We also include the following RBC moments as targets: (i) the standard deviations of aggregate output, investment and labor; (ii) the correlation with output for investment and labor; and (iii) the first-order autocorrelation of labor. These moments are computed as follows: consumption is real consumption per capita in the U.S. data excluding consumption on durable goods and investment is real investment per capita in the U.S. data including consumption on durable goods. Output is calculated as the sum of consumption and investment. The nominal data are from the National Income and Product Accounts provided by the U.S. Bureau of Economic Analysis. The nominal variables are adjusted by the GDP deflator. To scale by population, we use quarterly averages of the civilian noninstitutional population (CNP16OV) from the Federal Reserve Economic Data. Hours worked are measured by hours of all persons in the non-farm business sector (HOANBS) from the FRED. All series are log-detrended. For the RBC moments, we apply Hodrick-Prescott (HP) filter to the time series with a smoothing parameter of 1600 and compute the moments with the cyclical components. Table 1 summarizes the calibration values for the deep parameters and Column (1) in Table 2 presents the moments in the data.

Table 1: Calibrated Parameter Values

Parameter	Value	Description
	Parameters Calibrated from Existing Literature	
β	0.99	Discount factor
α	0.30	Capital share in production
ν	1.87	Inverse Frisch elasticity of labor supply
φ	39.80	Weight on labor disutility
δ	0.025	Capital depreciation rate
θ	7.5	Elasticity of substitution of intermediate goods
	Parameters Calibrated by Matching Moments	
ρ_a	0.970	Persistence of technology shock
σ_a	0.0074	Std of technology shock
ρ_z	0.907	Persistence of sentiment shock
σ_z	0.0052	Std of sentiment shock
σ_ε	0.497	Std of idiosyncratic demand shocks
σ_v	0.233	Std of noise
λ	0.105	Weight on idiosyncratic demand shock in signal
γ	0.809	Decay rate of noise

4.2. Dynamics

Based on the calibrated model, we quantitatively document the dynamic impacts of the sentiment shock on the aggregate economy. Figure 2 reports the responses of aggregate variables to a one-standard-deviation positive sentiment shock. This figure shows that the sentiment shock mimics a technology shock in generating comovements among key macro variables. That is, a positive sentiment shock boosts the aggregate economy by increasing output, labor, investment, and consumption, which confirms our analysis in Section 3.

The intuition proceeds as follows. When households become more optimistic, reflected by a positive sentiment shock, the signal s_{jt} received by an individual firm increases. The firm partially attributes this favorable signal to an increase in demand based on the signal extraction problem. The positive sentiment coordinates all intermediate goods firms' responses of production decisions, shifting out the supply curves of intermediate goods. Given that the sentiment shock is persistent over time, the optimism increases the expected future marginal return on capital and then stimulates firms' investments. When the goods markets open, the increase in the supply of intermediate goods raises the supply of final goods and depresses the price level. In the REE, the rise in the supply confirms the rise in the demand driven by the positive sentiment. At the same time, the decrease of the price level increases the real wage, rationalizing the rise in the labor supply in the first subperiod, and the labor market clears.

To see the role of incomplete information on the history of sentiments, we also plot the impulse responses to a positive sentiment shock when the sentiment process z_t is i.i.d over time. These responses are represented by the dashed lines and are very short-lived in this case. Following a positive sentiment shock, the responses of output, investment, and labor are strong in the impact period but quickly die out. The response of consumption lasts for multiple periods through capital accumulation.

The above analysis reveals that a nonfundamental sentiment shock can amplify business cycle fluctuations, which is echoed by the moments reported in Table 2. Column (1) stands for the moments in the U.S. data, Column (2) for the moments generated from the sentiment-driven equilibrium, and Column (3) for those from the fundamental equilibrium. In the sentiment-driven equilibrium where both technology and sentiment shocks are present, output volatility is 1.650%, which is close to the data. When shutting down the sentiment shocks, the output volatility drops by 31 percentage points and becomes as low as 1.132%. Moreover, the sentiment-driven equilibrium implies volatility of the labor market close to the data. In contrast, the fundamental equilibrium (a typical RBC model) does poorly at generating a reasonable value of this volatility. This result is intuitive: the sentiment shocks coordinate firms' production decisions and thus fluctuate the aggregate demand for labor, increasing the labor volatility.

Comparing Columns (1) and (2) shows that our model-generated moments fit the data reasonably well. For the purpose of theoretical illustration, our model builds on a standard RBC model and abstracts away from many common modeling features which could help the quan-

Table 2: Business Cycle Moments

	Data	Model	
		Sentiment	Fundamental
	(1)	(2)	(3)
		Volatility (%)	
output	1.627	1.650	1.132
consumption	0.720	1.200	0.529
investment	4.423	4.428	3.927
labor	1.639	1.710	0.223
		Correlation with Output	
consumption	0.761	0.932	0.962
investment	0.927	0.901	0.987
labor	0.767	0.802	0.974
		Auto-correlation	
output	0.901	0.735	0.730
consumption	0.810	0.689	0.778
investment	0.911	0.758	0.717
labor	0.941	0.731	0.715
		Correlation with 3-Quarter-Ahead Forecast	
output	0.947	0.913	0.921
investment	0.945	0.840	0.736
		Response to a S.D. Sentiment Shock	
output	appr. 1.1%	0.85%	0%

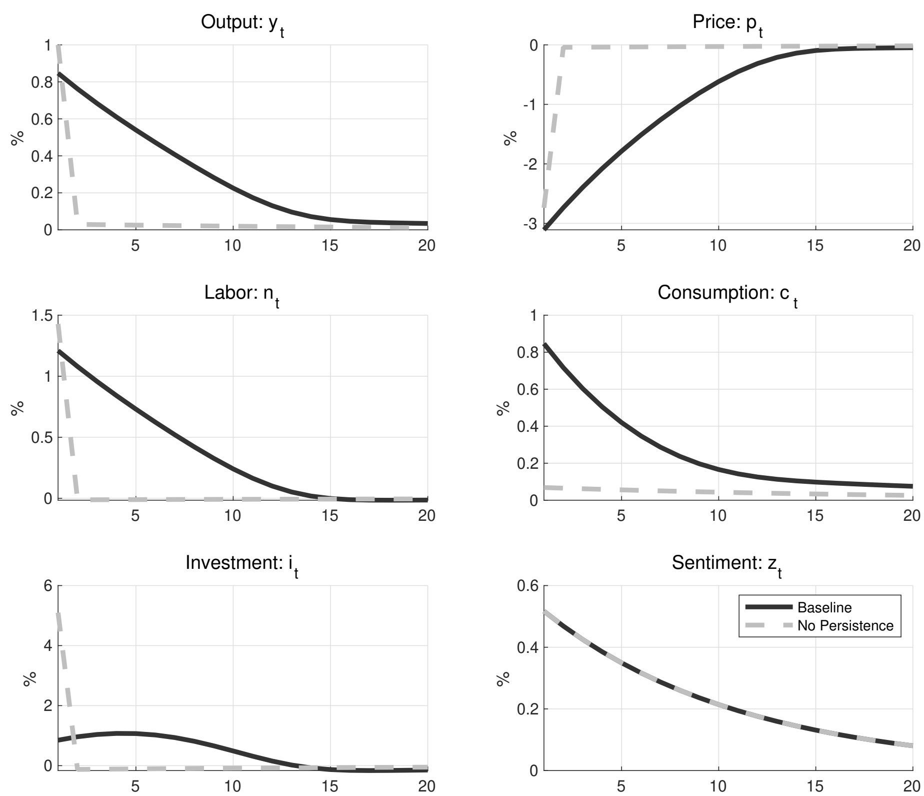
Notes: Column (1) summarizes the moments in the U.S. data. Column (2) is for the sentiment-driven equilibrium where both technology shocks and sentiment shocks present. Column (3) is for the fundamental equilibrium where only technology shocks present. Columns (2) and (3) are based on the same calibrated values for deep parameters. To compute volatility, correlation with output and auto-correlation, all series are HP filtered with a smoothing parameter at 1600.

titative performance. The results in Table 2 demonstrate the significant potential of sentiment shocks in accounting for real business cycles.

5. Conclusion

This paper studies an otherwise standard real business cycle model in which firms face incomplete information about their exact demand when making their production decisions. We find that the equilibrium outcome can be influenced by sentiments that are unrelated to fundamentals, even though all agents are rational. The underlying rationale is that sentiments in firms' signals can affect agents' forecasts on the equilibrium output and price level and hence their best responses to the signals. Such coordination gives rise to a self-fulfilling REE, which can exist with common types of preferences and is different from the fundamental equilibrium

Figure 2: Dynamic Impacts of the Sentiment Shock on the Aggregate Economy



Notes: This figure reports the impulse responses of aggregate variables to a one-standard-deviation positive sentiment shock. The vertical axis indicates the percentage deviation of one particular variable from its steady state in the sentiment-driven equilibrium. The solid lines are responses in the full-fledged model with incomplete information on the history of sentiments, as described in Section 3.3. The dashed lines are responses in the model with a complete information set on the history of sentiment process.

under complete information. We then calibrate the model with sentiment equilibrium based on U.S. aggregate data. The quantitative results show that pure sentiment shocks cause fluctuations in output, consumption, investment, and labor, and the comovements among them. The model-implied dynamics explain the empirical observations reasonably well, suggesting a nonnegligible role in accounting for business cycles. By further introducing an incomplete information structure into the history of sentiments, the model can produce persistent aggregate responses to sentiment shocks.

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Appendix A. Proposition Proofs

A.1. Proof of Proposition 1

We first derive the policy function of the price level. From the labor supply condition (15) and $W_t = 1$, we have $P_t = \frac{1}{\varphi} N_t^{-v}$. Substituting N_t with (18) yields

$$P_t = \varphi^{-\frac{\alpha}{\alpha+v}} \left[\frac{(\theta-1)(1-\alpha)}{\theta} \right]^{-\frac{\nu}{\alpha+\nu}} \mu^{-\frac{\nu}{\nu+\alpha}\frac{1-\alpha+\alpha\theta}{\theta-1}} A_t^{-\frac{\nu}{\alpha+\nu}} K_{t-1}^{-\frac{\alpha\nu}{\alpha+\nu}}. \tag{A.1}$$

From (18) and (19), we can derive the policy function of labor as

$$N_t = \varphi^{-\frac{1}{\alpha+\nu}} \left[\frac{\theta-1}{\theta}(1-\alpha) \right]^{\frac{1}{\alpha+\nu}} \mu^{\frac{1}{\alpha+\nu}\frac{1-\alpha+\alpha\theta}{\theta-1}} A_t^{\frac{1}{\alpha+\nu}} K_{t-1}^{\frac{\alpha}{\alpha+\nu}}. \tag{A.2}$$

From (19), we immediately have the policy function of output

$$Y_t = \varphi^{-\frac{1-\alpha}{\alpha+v}} \left[\frac{\theta-1}{\theta}(1-\alpha) \right]^{\frac{1-\alpha}{\alpha+v}} \mu^{\frac{1+\nu}{\alpha+\nu}\frac{1-\alpha+\alpha\theta}{\theta-1}} A_t^{\frac{1+v}{\alpha+v}} K_{t-1}^{\frac{\alpha(1+v)}{\alpha+v}}. \tag{A.3}$$

Taking logarithm on both sides of (A.3) yields

$$y_t^{\mathbf{f}} = \mathbf{G}_t^{\mathbf{f}}(a_t, k_t) \equiv \Xi_y^{\mathbf{f}}(\mu, \alpha, v, \theta, \varphi) + \mathbf{\Lambda}_y^{\mathbf{f}}(\alpha, v)(a_t + \alpha k_{t-1}), \tag{A.4}$$

where $\Xi_y^{\mathbf{f}}(\mu, \alpha, v, \theta, \varphi) = \log\left\{ \varphi^{-\frac{1-\alpha}{\alpha+v}} \left[\frac{(\theta-1)(1-\alpha)}{\theta} \right]^{\frac{1-\alpha}{\alpha+v}} \mu^{\frac{1+\nu}{\alpha+\nu}\frac{1-\alpha+\alpha\theta}{\theta-1}} \right\}$ and $\mathbf{\Lambda}_y^{\mathbf{f}}(\alpha, v) = \frac{1+v}{\alpha+v}$.

Taking logarithm on both sides of (A.1) yields

$$p_t^{\mathbf{f}} = \Xi_p^{\mathbf{f}}(\mu, \alpha, v, \theta, \varphi) + \mathbf{\Lambda}_p^{\mathbf{f}}(\alpha, v)(a_t + \alpha k_{t-1}), \tag{A.5}$$

where $\Xi_p^{\mathbf{f}}(\mu, \alpha, v, \theta, \varphi) = \log\left\{ \varphi^{-\frac{\alpha}{\alpha+v}} \left[\frac{(\theta-1)(1-\alpha)}{\theta} \right]^{-\frac{\nu}{\alpha+\nu}} \mu^{-\frac{\nu}{\nu+\alpha}\frac{1-\alpha+\alpha\theta}{\theta-1}} \right\}$ and $\mathbf{\Lambda}_p^{\mathbf{f}}(\alpha, v) = -\frac{\nu}{\alpha+\nu}$.

Using equations (5), (7) and (16), we could express the profits of intermediate goods producer j in the fundamental equilibrium $\Pi_{jt}\left(\epsilon_{jt}, K_{jt-1}; a_t, K_{t-1}\right)$ as

$$\begin{aligned} \Pi_{jt}\left(\epsilon_{jt}, K_{jt-1}\right) &= P_{jt}Y_{jt} - N_{jt} \\ &= P_t \epsilon_{jt}^{\frac{1}{\theta}} Y_t^{\frac{1}{\theta}} Y_{jt}^{1-\frac{1}{\theta}} - \left(\frac{Y_{jt}}{A_t K_{jt-1}^{\alpha}} \right)^{\frac{1}{1-\alpha}} \\ &= \kappa_\pi \epsilon_{jt}^{\frac{1}{1+\alpha(\theta-1)}} P_t^{\frac{\theta}{1+\alpha(\theta-1)}} Y_t^{\frac{1}{1+\alpha(\theta-1)}} A_t^{\frac{\theta-1}{1+\alpha(\theta-1)}} K_{jt-1}^{\frac{\alpha(\theta-1)}{1+\alpha(\theta-1)}}, \end{aligned} \tag{A.6}$$

where $\kappa_\pi = \left[\frac{(\theta-1)(1-\alpha)}{\theta} \right]^{\frac{(1-\alpha)(\theta-1)}{1-\alpha+\alpha\theta}} \frac{1+(\theta-1)\alpha}{\theta}$; and P_t and Y_t are given by (A.1) and (A.3).

Then we can write the intertemporal Euler equation for investment decision as

$$1 = \beta \mathbb{E}_t \left\{ \frac{U_C\left(C_{t+1}^e, N_t\right)}{U_C\left(C_t^e, N_t\right)} \left[\frac{1}{P_{t+1}} \frac{\partial \Pi_{t+1}\left(\epsilon_{jt+1}, K_{jt}\right)}{\partial K_{jt}} + (1-\delta) \right] \right\}. \tag{A.7}$$

where $\frac{\partial \Pi_{t+1}\left(\epsilon_{jt+1}, K_{jt}\right)}{\partial K_{jt}} = \frac{\alpha(\theta-1)}{\theta} \left[\frac{(\theta-1)(1-\alpha)}{\theta}\right]^{\frac{(1-\alpha)(\theta-1)}{1-\alpha+\alpha\theta}} \epsilon_{jt+1}^{\frac{1}{1+\alpha(\theta-1)}} P_{t+1}^{\frac{\theta}{1+\alpha(\theta-1)}} Y_{t+1}^{\frac{1}{1+\alpha(\theta-1)}} A_{t+1}^{\frac{\theta-1}{1+\alpha(\theta-1)}} K_{jt}^{-\frac{1}{1+\alpha(\theta-1)}}$. The optimal condition indicates that K_{jt} only depends on the aggregate states $\{a_t, K_{t-1}\}$ and thus $K_{jt} = K_t$ for all $j \in [0,1]$. Given the dynamics of capital, consumption C_t can be residually solved from the resource constraint (14).

A.2. Proof of Proposition 2

A.2.1. An Intermediate Goods Firm's Problem

Analogous to the fundamental equilibrium, we first solve the optimal production decision faced by an intermediate goods firm j. It is essentially an information extraction problem. In particular, substituting (5), (7) into (9) yields

$$\Pi_t\left(K_{jt-1}, s_{jt}\right) = \max_{\{Y_{jt}\}} \mathbb{E}\left[P_t \epsilon_{jt}^{\frac{1}{\theta}} Y_t^{\frac{1}{\theta}} Y_{jt}^{1-\frac{1}{\theta}} - \left(\frac{Y_{jt}}{A_t K_{jt-1}^{\alpha}}\right)^{\frac{1}{1-\alpha}} | s_{jt} \right]. \tag{A.8}$$

The first-order condition with respect to Y_{jt} is

$$Y_{jt}^{\varrho} = \left(1 - \frac{1}{\theta}\right)(1-\alpha) A_t^{\frac{1}{1-\alpha}} K_{jt-1}^{\frac{\alpha}{1-\alpha}} \mathbb{E}\left(\epsilon_{jt}^{\frac{1}{\theta}} P_t Y_t^{\frac{1}{\theta}} | s_{jt}\right), \tag{A.9}$$

where $\varrho = \frac{1}{\theta} + \frac{\alpha}{1-\alpha}$.

We define x_t as the logarithm of a variable X_t, i.e., $x_t \equiv \log X_t$. With signals $s_{jt} = \lambda \varepsilon_{jt} + (1-\lambda) z_t$, intermediate goods firms conjecture that the price level and aggregate output jointly follow

$$\begin{bmatrix} p_t \\ y_t \end{bmatrix} = \mathbf{\Xi^s} + \mathbf{\Lambda^s}\left(a_t + \alpha k_{t-1}\right) + \mathbf{\Theta^s} z_t, \tag{A.10}$$

where we normalize the second element of $\mathbf{\Theta^s}$ to be 1, implying that movements in y_t one-to-one correspond to movements in z_t. The other elements in the coefficient matrices $\{\mathbf{\Xi^s}, \mathbf{\Lambda^s}, \mathbf{\Theta^s}\}$ need to be determined.

Define $x_{jt} = \frac{1}{\theta}\varepsilon_{jt} + \omega \mathbf{\Theta^s} z_t$, where $\omega = \left[1, \frac{1}{\theta}\right]$. With the above conjecture, an individual firm j's expectation conditional on the signal s_{jt} can be expressed as

$$\begin{aligned} & \mathbb{E}\left(\epsilon_{jt}^{\frac{1}{\theta}} P_t Y_t^{\frac{1}{\theta}} | s_{jt}\right) \\ = \; & \exp\left[\omega \Xi^{\mathbf{s}} + \omega \Lambda^{\mathbf{s}}\left(a_t + \alpha k_{t-1}\right)\right] \mathbb{E}\left[\exp\left(x_{jt}\right) | s_{jt}\right] \\ = \; & \exp\left[\omega \Xi^{\mathbf{s}} + \omega \Lambda^{\mathbf{s}}\left(a_t + \alpha k_{t-1}\right)\right] \exp\left[\mathbb{E}\left(x_{jt} | s_{jt}\right) + \frac{1}{2}\mathbf{Var}\left(x_{jt} | s_{jt}\right)\right]. \end{aligned} \tag{A.11}$$

The second line is obtained by taking the constant terms $\mathbf{\Xi^s}$ and the fundamental $a_t + \alpha k_{t-1}$ out of the expectation operator as these terms are known to the firm. The third line is obtained with $x_{jt}|s_{jt}$ following a joint normal distribution.

The conditional expectation on $\mathbf{x}_{jt}$ satisfies

$$\mathbb{E}\left(x_{jt}|s_{jt}\right) = \Upsilon_{\mathbf{x}} s_{jt}, \tag{A.12}$$

where the coefficient $\Upsilon_{\mathbf{x}} = \frac{\frac{\lambda}{\theta}\sigma_\varepsilon^2 + (1-\lambda)\omega\boldsymbol{\Theta}\sigma_z^2}{\lambda^2\sigma_\varepsilon^2+(1-\lambda)^2\sigma_z^2}$ indicates the signal-noise ratio. The conditional variance of x_{jt} is

$$\mathbf{Var}\left(x_{jt}|s_{jt}\right) = \frac{1}{\theta^2}\sigma_\varepsilon^2 + \left(\omega\boldsymbol{\Theta}^{\mathbf{s}}\right)^2\sigma_z^2 - \frac{\left[\frac{\lambda}{\theta}\sigma_\varepsilon^2 + (1-\lambda)\omega\boldsymbol{\Theta}^{\mathbf{s}}\sigma_z^2\right]^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2}. \tag{A.13}$$

Then from (A.11), we can express the production of intermediate goods producer j as

$$y_{jt} = \bar{y} + \frac{1}{\varrho}\frac{1}{1-\alpha}\left(a_t + \alpha k_{jt-1}\right) + \frac{1}{\varrho}\omega\boldsymbol{\Lambda}^{\mathbf{s}}\left(a_t + \alpha k_{t-1}\right) + \frac{1}{\varrho}\Upsilon_{\mathbf{x}} s_{jt}, \tag{A.14}$$

where the constant term satisfies $\bar{y} = \frac{1}{\varrho}\left[\log\left(1-\frac{1}{\theta}\right)(1-\alpha)\right] + \frac{1}{\varrho}\omega\boldsymbol{\Xi}^{\mathbf{s}} + \frac{1}{2\varrho}\mathbf{Var}\left(x_{jt}|s_{jt}\right)$. This best response function shows that the firm j's optimal production decision depends on its own capital stock k_{jt-1}, the aggregate state $a_t + \alpha k_{t-1}$, and the signal s_{jt}.

With (A.14), profits of an intermediate goods producer are given by

$$\Pi_t\left(K_{jt-1}, s_{jt}\right) = \varpi^\pi\left[\mathbb{E}\left(\epsilon_{jt}^{\frac{1}{\theta}} P_t Y_t^{\frac{1}{\theta}}|s_{jt}\right)\right]^{\frac{1}{\varrho}\frac{1}{1-\alpha}} A_t^{\frac{1}{1-\alpha}\frac{1}{\varrho}\frac{\theta-1}{\theta}} K_{jt-1}^{\frac{\alpha}{1-\alpha}\frac{1}{\varrho}\frac{\theta-1}{\theta}}. \tag{A.15}$$

where $\varpi^\pi = \frac{1+\alpha(\theta-1)}{\theta}\left[\left(1-\frac{1}{\theta}\right)(1-\alpha)\right]^{\frac{1}{\varrho}\frac{\theta-1}{\theta}}$. Then the optimal investment decision is implicitly determined by

$$1 = \beta\mathbb{E}_t\frac{\Lambda_{t+1}}{\Lambda_t}\left[\frac{1}{P_{t+1}}\frac{\partial\Pi\left(K_{jt}, s_{jt+1}; a_{t+1}, K_t\right)}{\partial K_{jt}} + (1-\delta)\right], \tag{A.16}$$

where $\frac{\partial\Pi_{t+1}\left(K_{jt}, s_{jt+1}\right)}{\partial K_{jt}} = \frac{\alpha(\theta-1)}{1+\alpha(\theta-1)}\varpi^\pi\left[\mathbb{E}\left(\epsilon_{jt+1}^{\frac{1}{\theta}} P_{t+1} Y_{t+1}^{\frac{1}{\theta}}|s_{jt+1}\right)\right]^{\frac{\theta}{1+\alpha(\theta-1)}} A_{t+1}^{\frac{\theta-1}{1+\alpha(\theta-1)}} K_{jt}^{-\frac{1}{1+\alpha(\theta-1)}}$.
Since the components $\left\{\varepsilon_{jt}, z_t\right\}$ in the signal s_{jt} are independent across periods, the expectation of $\left[\frac{\partial\Pi_{t+1}\left(K_{jt}, s_{jt+1}\right)}{\partial K_{jt}}\right]$ only depends on the aggregate states $\{a_t, K_{t-1}\}$ and the individual capital stock K_{jt}. As a result, the intertemporal Euler equation implies that the optimal investment decision only depends on the aggregate states, i.e., $K_{jt} = K_t$ for all individual firm j.

A.2.2. Sentiment-Driven Equilibrium

We are now ready to solve the sentiment-driven equilibrium. Aggregating the individual decision y_{jt} according to the CES production function, we have

$$y_t = \frac{\theta}{\theta-1}\log\left[\int_0^1 \exp\left(\frac{1}{\theta}\varepsilon_{jt} + \frac{\theta-1}{\theta}y_{jt}\right)dj\right]. \tag{A.17}$$

Substituting y_{jt} with (A.14) yields

$$y_t = \bar{y} + \frac{\theta}{\theta-1}\kappa_y^2\frac{\sigma_\varepsilon^2}{2} + \frac{1}{\varrho}\left(\frac{1}{1-\alpha} + \omega\mathbf{\Lambda}^{\mathbf{s}}\right)(a_t + \alpha k_{t-1}) + \frac{1}{\varrho}\Upsilon_{\mathbf{x}}(1-\lambda)z_t, \tag{A.18}$$

where $\kappa_y = \frac{1}{\theta} + \frac{\theta-1}{\theta}\frac{1}{\varrho}\Upsilon_{\mathbf{x}}\lambda$.

Let $\mathbf{\Xi}_y^{\mathbf{s}}$ and $\mathbf{\Lambda}_y^{\mathbf{s}}$ denote the second element in $\mathbf{\Xi}^{\mathbf{s}}$ and $\mathbf{\Lambda}^{\mathbf{s}}$, respectively. Matching coefficients in last equation with those in the conjecture (A.10), we have

$$\mathbf{\Xi}_y^{\mathbf{s}} = \bar{y} + \frac{\theta}{\theta-1}\frac{\kappa_y^2}{2}\sigma_\varepsilon^2, \tag{A.19}$$

$$\mathbf{\Lambda}_y^{\mathbf{s}} = \frac{1}{\varrho}\left(\frac{1}{1-\alpha} + \omega\mathbf{\Lambda}^{\mathbf{s}}\right), \tag{A.20}$$

$$1 = \frac{1}{\varrho}\Upsilon_{\mathbf{x}}(1-\lambda). \tag{A.21}$$

From production function (7), we can derive the labor demand of firm j as

$$n_{jt} = \frac{1}{1-\alpha}\left(y_{jt} - a_t - \alpha k_{jt-1}\right). \tag{A.22}$$

Substituting the optimal production y_{jt} with (A.14) yields

$$n_{jt} = \frac{1}{1-\alpha}\bar{y} + \frac{1}{1-\alpha}\left(\frac{1}{\varrho}\frac{1}{1-\alpha} + \frac{1}{\varrho}\omega\mathbf{\Lambda}^{\mathbf{s}} - 1\right)(a_t + \alpha k_{t-1}) + \frac{1}{\varrho}\frac{1}{1-\alpha}\Upsilon_{\mathbf{x}}s_{jt}. \tag{A.23}$$

Aggregate labor is thus given by

$$\begin{aligned} n_t &= \log\int_0^1 \exp\left(n_{jt}\right)dj \\ &= \frac{1}{1-\alpha}\bar{y} + \frac{\kappa_n^2}{2}\sigma_\varepsilon^2 + \frac{1}{1-\alpha}\left(\frac{1}{\varrho}\frac{1}{1-\alpha} + \frac{1}{\varrho}\omega\mathbf{\Lambda}^{\mathbf{s}} - 1\right)(a_t + \alpha k_{t-1}) + \frac{1}{\varrho}\frac{1}{1-\alpha}\Upsilon_{\mathbf{x}}(1-\lambda)z_t, \end{aligned} \tag{A.24}$$

where $\kappa_n = \frac{\lambda}{1-\alpha}\frac{\Upsilon_{\mathbf{x}}}{\varrho}$.

Log-linearizing the labor supply curve (15), we obtain

$$\begin{aligned} p_t &= -\log\varphi - v n_t \\ &= -\log\varphi - \frac{v}{1-\alpha}\bar{y} - v\frac{\kappa_n^2}{2}\sigma_\varepsilon^2 - \frac{v}{1-\alpha}\left(\frac{1}{\varrho}\frac{1}{1-\alpha} + \frac{1}{\varrho}\omega\mathbf{\Lambda}^{\mathbf{s}} - 1\right)(a_t + \alpha k_{t-1}) - \frac{v}{1-\alpha}\frac{1}{\varrho}\Upsilon_{\mathbf{x}}(1-\lambda)z_t. \end{aligned} \tag{A.25}$$

Let $\mathbf{\Theta}_p^{\mathbf{s}}$, $\mathbf{\Xi}_p^{\mathbf{s}}$ and $\mathbf{\Lambda}_p^{\mathbf{s}}$ denote the first element in $\mathbf{\Theta}^{\mathbf{s}}$, $\mathbf{\Xi}^{\mathbf{s}}$, and $\mathbf{\Lambda}^{\mathbf{s}}$, respectively. Matching the coefficients in the last equation with those in (A.10) yields

$$\Xi_p^{\mathbf{s}} = -\log\varphi - \frac{v}{1-\alpha}\bar{y} - v\frac{\kappa_n^2}{2}\sigma_\varepsilon^2, \tag{A.26}$$

$$\mathbf{\Lambda}_p^{\mathbf{s}} = -\frac{v}{1-\alpha}\left(\frac{1}{\varrho}\frac{1}{1-\alpha} + \frac{1}{\varrho}\omega\mathbf{\Lambda}^{\mathbf{s}} - 1\right), \tag{A.27}$$

$$\mathbf{\Theta}_p^{\mathbf{s}} = -\frac{v}{1-\alpha}\frac{1}{\varrho}\Upsilon_{\mathbf{x}}(1-\lambda). \tag{A.28}$$

In summary, equations (A.19)-(A.21) and (A.26)-(A.28) jointly determine all the elements in $\Xi^{\mathbf{s}}$, $\mathbf{\Lambda}^{\mathbf{s}}$ and $\mathbf{\Theta}^{\mathbf{s}}$. It can be shown that $\mathbf{\Lambda}^{\mathbf{s}}(\alpha, v) = \begin{bmatrix} -\frac{v}{\alpha+v} \\ \frac{1+v}{\alpha+v} \end{bmatrix}$ and $\mathbf{\Theta}^{\mathbf{s}} = \begin{bmatrix} -\frac{v}{1-\alpha} \\ 1 \end{bmatrix}$.

After solving $\Xi^{\mathbf{s}}$, $\mathbf{\Lambda}^{\mathbf{s}}$ and $\mathbf{\Theta}^{\mathbf{s}}$, we can pin down the variance of sentiment shocks, σ_z^2, from equation (A.21),

$$\sigma_z^2 = \frac{1-\alpha}{(\alpha+\nu)(1-\lambda)^2}\left(\frac{1-2\lambda}{\theta} - \frac{\alpha}{1-\alpha}\lambda\right)\lambda\sigma_\varepsilon^2. \tag{A.29}$$

Noticing the fact that $\mathbf{\Lambda}_y^{\mathbf{s}}(\alpha, v) = \mathbf{\Lambda}_y^{\mathbf{f}}(\alpha, v) = \frac{1+v}{\alpha+v}$, we can rewrite the policy function of y_t in the sentiment-driven equilibrium as

$$y_t^{\mathbf{s}} = \mathbf{G}_t^{\mathbf{s}}(a_t, k_{t-1}, z_t) = \Xi_y^{\mathbf{s}} - \Xi_y^{\mathbf{f}} + \mathbf{G}_t^{\mathbf{f}}(a_t, k_{t-1}) + z_t, \tag{A.30}$$

where $\mathbf{G}_t^{\mathbf{f}}(a_t, k_{t-1})$ is the policy function of y_t in the fundamental equilibrium and $\Xi_y^{\mathbf{s}} - \Xi_y^{\mathbf{f}}$ is a constant. This completes the proof of Proposition 2.

A.3. Stability Under Learning

There are two fixed-point solutions to the mapping $\sigma_{zt+1} = h(\sigma_{zt})$: (i) $\sigma_z = 0$ which corresponds to the fundamental equilibrium; (ii) σ_z given by (A.29) which corresponds to the sentiment-driven equilibrium.

To check the stability of the two equilibria, we evaluate $h'(0)$ and $h'(\sigma_z)$. Around the fundamental equilibrium,

$$h'(0) = 1 - g + g\frac{(1-\lambda)(1-\alpha)}{(1-\alpha+\alpha\theta)\lambda} > 1, \tag{A.31}$$

for $g \in (0,1)$ when $\frac{1-2\lambda}{\lambda} > \frac{\alpha}{1-\alpha}\theta$. It means that the fundamental equilibrium in unstable under learning.

Around the sentiment-driven equilibrium,

$$h'(\sigma_z) = 1 - 2g\sigma_z^2\frac{(1-\lambda)^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2}\frac{\alpha\theta - \theta(1-\alpha)\mathbf{\Theta}_p^{\mathbf{s}}}{1-\alpha+\alpha\theta}. \tag{A.32}$$

When the gain g is sufficiently small, $|h'(\sigma_z)| < 1$. This implies that the sentiment-driven equilibrium is stable under learning by the E-stability principle (Evans and Honkapohja, 2012).

Appendix B. Equilibria with a More General Utility Function

In this section, we solve the fundamental and sentiment-driven equilibrium with a more general utility function $U(C_t, N_t)$. Without loss of generality, we consider KPR preferences with a utility function

$U(C_t, N_t) = \log(C_t) - \varphi \frac{N_t^{1+\nu}}{1+\nu}$. The household's labor supply decision becomes

$$\frac{U_c(C_t^e, N_t)}{P_t^e} = \varphi N_t^{\nu}. \tag{B.1}$$

Under such a general utility function, we cannot obtain analytical solutions as those in the GHH case. Therefore, we conduct the analysis based on their linearized dynamic systems. We start with the fundamental equilibrium.

B.1. Fundamental Equilibrium

The fundamental equilibrium system for $\{P_t, Y_t, N_t, C_t, K_t\}$ consists of Euler equation for investment (13), resource constraint (14), aggregate labor (18), aggregate output (19), and labor supply curve (B.1). Let $\hat{x}_t$ denote the percentage deviation of x_t from its steady-state value. The linearized system for $\left\{\hat{p}_t, \hat{y}_t, \hat{n}_t, \hat{k}_t, \hat{c}_t, a_t\right\}$ can be summarized as

$$\begin{aligned}
\hat{y}_t &= \frac{1}{\alpha} a_t + \hat{k}_{t-1} + \frac{1-\alpha}{\alpha} \hat{p}_t, \\
\hat{y}_t &= a_t + \alpha \hat{k}_{t-1} + (1-\alpha)\hat{n}_{t-1}, \\
\hat{y}_t &= \frac{C^{\mathbf{f}}}{Y^{\mathbf{f}}} \hat{c}_t + \frac{K^{\mathbf{f}}}{Y^{\mathbf{f}}} \hat{k}_t - (1-\delta) \frac{K^{\mathbf{f}}}{Y^{\mathbf{f}}} \hat{k}_{t-1}, \\
0 &= \hat{c}_t - \mathbb{E}_t \hat{c}_{t+1} + \frac{1-\beta(1-\delta)}{1-\alpha+\alpha\theta} \left[-\hat{k}_t + (\theta-1)\mathbb{E}_t a_{t+1} + (1-\alpha)(\theta-1)\mathbb{E}_t \hat{p}_{t+1} + \mathbb{E}_t \hat{y}_{t+1} \right], \\
v\hat{n}_t &= -\hat{c}_t - \hat{p}_t, \\
a_t &= \rho_a a_{t-1} + \varepsilon_{at},
\end{aligned}$$

where $C^{\mathbf{f}}$, $Y^{\mathbf{f}}$ and $K^{\mathbf{f}}$ are steady-state consumption, output and capital, respectively, in the fundamental equilibrium.

Let $\hat{\mathbf{X}}_t^{\mathbf{f}} = [\hat{p}_t, \hat{y}_t, \hat{n}_t, \hat{c}_t]'$ and $\hat{\mathbf{S}}_t^{\mathbf{f}} = \left[a_t, \hat{k}_{t-1}\right]'$. The above dynamic system can be expressed more compactly as

$$\mathbb{A}^{\mathbf{f}} \begin{bmatrix} \hat{\mathbf{S}}_t^{\mathbf{f}} \\ \hat{\mathbf{X}}_t^{\mathbf{f}} \end{bmatrix} = \mathbb{B}^{\mathbf{f}} \mathbb{E}_t \begin{bmatrix} \hat{\mathbf{S}}_{t+1}^{\mathbf{f}} \\ \hat{\mathbf{X}}_{t+1}^{\mathbf{f}} \end{bmatrix}, \tag{B.2}$$

where $\mathbb{A}^{\mathbf{f}}$ and $\mathbb{B}^{\mathbf{f}}$ are matrices depending on deep parameters and the steady-state values. This system is essentially the log-linearized version of a standard RBC model, which usually has a unique saddle path that satisfies

$$\begin{bmatrix} \hat{\mathbf{X}}_t^{\mathbf{f}} \\ \hat{k}_t^{\mathbf{f}} \end{bmatrix} = \mathbf{\Lambda}^{\mathbf{f}} \hat{\mathbf{S}}_t^{\mathbf{f}}, \tag{B.3}$$

where $\mathbf{\Lambda}^{\mathbf{f}}$ is a coefficient matrix obtained from the standard procedure of solving a RBC model.

B.2. Sentiment-Driven Equilibrium

We now solve the sentiment-driven equilibrium with the guess-and-verify approach. Intermediate goods producers set their beliefs on the process of aggregate control variables $\hat{\mathbf{X}}_t^{\mathbf{s}} = [\hat{p}_t, \hat{y}_t, \hat{n}_t, \hat{c}_t]'$ to follow

$$\begin{bmatrix} \hat{\mathbf{X}}_t^{\mathbf{s}} \\ \hat{k}_t^{\mathbf{s}} \end{bmatrix} = \mathbf{\Lambda}^{\mathbf{s}} \hat{\mathbf{S}}_t^{\mathbf{s}} + \mathbf{\Theta}^{\mathbf{s}} z_t, \tag{B.4}$$

where $\hat{\mathbf{S}}_t^{\mathbf{s}} = \left[a_t, \hat{k}_{t-1}\right]'$; z_t is the sentiment shock; $\mathbf{\Lambda}^{\mathbf{s}} = \begin{bmatrix} \mathbf{\Lambda}_{pa}^{\mathbf{s}} & \mathbf{\Lambda}_{pk}^{\mathbf{s}} \\ \mathbf{\Lambda}_{ya}^{\mathbf{s}} & \mathbf{\Lambda}_{yk}^{\mathbf{s}} \\ \mathbf{\Lambda}_{na}^{\mathbf{s}} & \mathbf{\Lambda}_{nk}^{\mathbf{s}} \\ \mathbf{\Lambda}_{ca}^{\mathbf{s}} & \mathbf{\Lambda}_{ck}^{\mathbf{s}} \\ \mathbf{\Lambda}_{ka}^{\mathbf{s}} & \mathbf{\Lambda}_{kk}^{\mathbf{s}} \end{bmatrix}$ and $\mathbf{\Theta}^{\mathbf{s}} = \begin{bmatrix} \mathbf{\Theta}_p^{\mathbf{s}} \\ \mathbf{\Theta}_y^{\mathbf{s}} \\ \mathbf{\Theta}_n^{\mathbf{s}} \\ \mathbf{\Theta}_c^{\mathbf{s}} \\ \mathbf{\Theta}_k^{\mathbf{s}} \end{bmatrix}$ are coefficient matrices to be determined. Here, we normalize $\mathbf{\Theta}_y = 1$ as before.

Under the above forecast rule, the optimal labor demand and production decision imply that $\left\{n_{jt}, y_{jt}\right\}$ are linear functions of its own fundamental, $\left[a_t, k_{jt-1}\right]'$, and its forecast on the aggregate economy conditional on the signal it receives, $\mathbb{E}\left(\hat{\mathbf{X}}_t^{\mathbf{s}}|s_{jt}\right)$.

To see this, we start from deriving the conditional expectation term in the optimal decision of Y_{jt}, $\mathbb{E}\left(\epsilon_{jt}^{\frac{1}{\theta}} P_t Y_t^{\frac{1}{\theta}}|s_{jt}\right)$, which can be written as

$$\begin{aligned}\mathbb{E}\left(\epsilon_{jt}^{\frac{1}{\theta}} P_t Y_t^{\frac{1}{\theta}}|s_{jt}\right) =& P^{\mathbf{s}}\left(Y^{\mathbf{s}}\right)^{\frac{1}{\theta}} \exp\left[\frac{1}{2}\mathbf{Var}\left(\mathbf{x}_t|s_{jt}\right)\right] \\ & \exp\left[\left(\mathbf{\Lambda}_{pa}^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Lambda}_{ya}^{\mathbf{s}}\right)a_t + \left(\mathbf{\Lambda}_{pk}^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Lambda}_{yk}^{\mathbf{s}}\right)\hat{k}_{t-1}\right]\exp\left[\mathbb{E}\left(x_{jt}|s_{jt}\right)\right]\end{aligned} \tag{B.5}$$

where $x_{jt} = \frac{1}{\theta}\varepsilon_{jt} + \left(\frac{1}{\theta} + \mathbf{\Theta}_p^{\mathbf{s}}\right)z_t$, $P^{\mathbf{s}}$ and $Y^{\mathbf{s}}$ are steady-state price and output in the sentiment-driven equilibrium. The signal extraction problem implies $\mathbb{E}\left(x_{jt}|s_{jt}\right) = \Upsilon_{\mathbf{x}} s_{jt}$, where $\Upsilon_{\mathbf{x}} = \frac{\frac{\lambda}{\theta}\sigma_\varepsilon^2 + \left(\frac{1}{\theta} + \mathbf{\Theta}_p^{\mathbf{s}}\right)(1-\lambda)\sigma_z^2}{\lambda^2\sigma_\varepsilon^2 + (1-\lambda)^2\sigma_z^2}$ is the signal-noise ratio. Substituting (B.5) into (10) yields the policy function of y_{jt} similar to that in the GHH case

$$y_{jt} = \bar{y} + \underbrace{\frac{1}{(1-\alpha)\varrho}\left(a_t + \alpha k_{jt-1}\right)}_{\text{individual fundamental}} + \underbrace{\frac{1}{\varrho}\left(\mathbf{\Lambda}_{pa}^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Lambda}_{ya}^{\mathbf{s}}\right)a_t + \frac{1}{\varrho}\left(\mathbf{\Lambda}_{pk}^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Lambda}_{yk}^{\mathbf{s}}\right)\hat{k}_{t-1} + \frac{1}{\varrho}\Upsilon_{\mathbf{x}} s_{jt}}_{\text{aggregate fundamental conditional on signal } s_{jt}}. \tag{B.6}$$

Aggregating $Y_{jt} = \exp\left(y_{jt}\right)$ gives aggregate output Y_t. Once we obtain Y_t, we can also derive the steady-state output $Y^{\mathbf{s}}$ in terms of $\mathbf{\Lambda}^{\mathbf{s}}$ and $\mathbf{\Theta}^{\mathbf{s}}$. Then we log-linearize Y_t around the steady state and obtain

$$\hat{y}_t = \frac{1}{(1-\alpha)\varrho}\left(a_t + \hat{k}_{t-1}\right) + \frac{1}{\varrho}\left(\mathbf{\Lambda}_{pa}^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Lambda}_{ya}^{\mathbf{s}}\right)a_t + \frac{1}{\varrho}\left(\mathbf{\Lambda}_{pk}^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Lambda}_{yk}^{\mathbf{s}}\right)\hat{k}_{t-1} + \frac{1-\lambda}{\varrho}\Upsilon_{\mathbf{x}} z_t. \tag{B.7}$$

Using the conjecture rules $\hat{y}_t = \mathbf{\Lambda}_{ya}^{\mathbf{s}} a_t + \mathbf{\Lambda}_{yk}^{\mathbf{s}}\hat{k}_{t-1} + z_t$ and $\hat{p}_t = \mathbf{\Lambda}_{pa}^{\mathbf{s}} a_t + \mathbf{\Lambda}_{pk}^{\mathbf{s}}\hat{k}_{t-1} + \mathbf{\Theta}_p^{\mathbf{s}} z_t$, the above equation can be rewritten as

$$\hat{y}_t = \frac{1}{\alpha}a_t + \hat{k}_{t-1} + \frac{1-\alpha}{\alpha}\hat{p}_t + \left(1 - \frac{1-\alpha}{\alpha}\mathbf{\Theta}_p^{\mathbf{s}}\right)z_t. \tag{B.8}$$

Equation (B.6) leads to that $Y_{jt} \propto \exp\left(\frac{\lambda}{\varrho}\Upsilon_{\mathbf{x}}\varepsilon_{jt}\right)$. Utilizing the fact that $K_{jt} = K_t$ for all $j \in [0,1]$, we have $N_{jt} \propto \exp\left(\frac{\lambda}{(1-\alpha)\varrho}\Upsilon_{\mathbf{x}}\varepsilon_{jt}\right)$. Given the labor market clearing condition $N_t = \int_0^1 N_{jt}dj$, we have

$$N_{jt} = \frac{\exp\left(\frac{\lambda}{(1-\alpha)\varrho}\Upsilon_{\mathbf{x}}\varepsilon_{jt}\right)}{\int_0^1 \exp\left(\frac{\lambda}{(1-\alpha)\varrho}\Upsilon_{\mathbf{x}}\varepsilon_{jt}\right)dj}N_t, \tag{B.9}$$

which is substituted into production function (7) of intermediate goods firms. Then substituting the resulted equation into production function (3) of final goods firms yields

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} \frac{\int_0^1 \exp\left[\left(\frac{1}{\theta} + \frac{\theta-1}{\theta}\frac{\lambda}{\varrho}\Upsilon_{\mathbf{x}}\right)\varepsilon_{jt}\right] dj}{\left[\int_0^1 \exp\left(\frac{\lambda}{(1-\alpha)\varrho}\Upsilon_{\mathbf{x}}\varepsilon_{jt}\right) dj\right]^{1-\alpha}}, \tag{B.10}$$

which yields the following log-linearized equation

$$\hat{y}_t = a_t + \alpha \hat{k}_{t-1} + (1-\alpha)\hat{n}_t. \tag{B.11}$$

We next log-linearize the labor supply curve and resource constraint and obtain

$$\hat{c}_t = -\hat{p}_t - \nu \hat{n}_t, \tag{B.12}$$

$$\hat{y}_t = \frac{C^{\mathbf{s}}}{Y^{\mathbf{s}}}\hat{c}_t + \frac{K^{\mathbf{s}}}{Y^{\mathbf{s}}}\hat{k}_t - (1-\delta)\frac{K^{\mathbf{s}}}{Y^{\mathbf{s}}}\hat{k}_{t-1}. \tag{B.13}$$

We derive the profit function analogously to (A.15) and obtain

$$\begin{aligned}\Pi_t(K_{jt-1}, s_{jt}) = \; & A_t^{\frac{\theta-1}{1-\alpha+\alpha\theta}} K_{jt-1}^{\frac{\alpha(\theta-1)}{1-\alpha+\alpha\theta}} \left\{ \left[(1-\alpha)\frac{\theta-1}{\theta}\right]^{\frac{\theta-1}{\theta}\frac{1}{\varrho}} \left(\epsilon_{jt}^{\frac{1}{\theta}} P_t Y_t^{\frac{1}{\theta}}\right) \left[\mathbb{E}_t\left(\epsilon_{jt}^{\frac{1}{\theta}} P_t Y_t^{\frac{1}{\theta}} | s_{jt}\right)\right]^{\frac{\theta-1}{\theta}\frac{1}{\varrho}} \right. \\ & \left. - \left[(1-\alpha)\frac{\theta-1}{\theta}\right]^{\frac{1}{1-\alpha}\frac{1}{\varrho}} \left[\mathbb{E}_t\left(\epsilon_{jt}^{\frac{1}{\theta}} P_t Y_t^{\frac{1}{\theta}} | s_{jt}\right)\right]^{\frac{1}{1-\alpha}\frac{1}{\varrho}} \right\}. \end{aligned} \tag{B.14}$$

Then for the Euler equation of investment, we have

$$1 = \beta \mathbb{E}_t \frac{C_t}{C_{t+1}} \left\{ \begin{array}{c} 1 - \delta + \Gamma_k \times A_{t+1}^{\frac{\theta-1}{1-\alpha+\alpha\theta}} K_t^{-\frac{1}{1-\alpha+\alpha\theta}} \times \\ \exp\left[\frac{\mathbf{\Lambda}_{ya}^{\mathbf{s}} + (\theta-1)(1-\alpha)\mathbf{\Lambda}_{pa}^{\mathbf{s}}}{1-\alpha+\alpha\theta} a_{t+1} + \frac{\mathbf{\Lambda}_{yk}^{\mathbf{s}} + (\theta-1)(1-\alpha)\mathbf{\Lambda}_{pk}^{\mathbf{s}}}{1-\alpha+\alpha\theta}\hat{k}_t\right] \end{array} \right\}.$$

where Γ_k absorbs all constants. Log-linearizing the above Euler equation around the steady state and making use of the conjecture rules $\hat{y}_t = \mathbf{\Lambda}_{ya}^{\mathbf{s}} a_t + \mathbf{\Lambda}_{yk}^{\mathbf{s}} \hat{k}_{t-1} + z_t$ and $\hat{p}_t = \mathbf{\Lambda}_{pa}^{\mathbf{s}} a_t + \mathbf{\Lambda}_{pk}^{\mathbf{s}} \hat{k}_{t-1} + \mathbf{\Theta}_p^{\mathbf{s}} z_t$, we obtain

$$0 = \hat{c}_t - \mathbb{E}_t \hat{c}_{t+1} + \frac{1-\beta(1-\delta)}{1-\alpha+\alpha\theta}\left[-\hat{k}_t + (\theta-1)\mathbb{E}_t a_{t+1} + (1-\alpha)(\theta-1)\mathbb{E}_t \hat{p}_{t+1} + \mathbb{E}_t \hat{y}_{t+1}\right], \tag{B.15}$$

where we utilize the assumption that z_t is i.i.d. over time, i.e., $\mathbb{E}_t z_{t+1} = 0$.

In summary, we have the linearized system (B.8), (B.11), (B.12), (B.13), and (B.15) for $\left\{\hat{p}_t, \hat{y}_t, \hat{n}_t, \hat{c}_t, \hat{k}_t\right\}$ in the sentiment-driven equilibrium. Meantime, the steady-state version of these five equations also constitutes a joint equation system for the steady-state values $\mathbf{X}^{\mathbf{s}} = [P^{\mathbf{s}}, Y^{\mathbf{s}}, N^{\mathbf{s}}, C^{\mathbf{s}}, K^{\mathbf{s}}]'$. Notice that the steady state also depend on $\mathbf{\Lambda}^{\mathbf{s}}$ and $\mathbf{\Theta}^{\mathbf{s}}$.

We can rewrite the dynamic system more compactly as

$$\begin{bmatrix} \hat{\mathbf{X}}_t^{\mathbf{s}} \\ \hat{k}_t^{\mathbf{s}} \end{bmatrix} = \mathbf{G}\left(\mathbf{\Lambda}^{\mathbf{s}}, \mathbf{\Theta}^{\mathbf{s}}, \mathbf{X}^{\mathbf{s}}\right) \begin{bmatrix} \hat{\mathbf{S}}_t^{\mathbf{s}} \\ z_t \end{bmatrix}. \tag{B.16}$$

Matching the coefficients in the conjecture rule (B.4) and the above policy function determines a unique solution of matrices $\mathbf{\Lambda}^{\mathbf{s}}$ and $\mathbf{\Theta}^{\mathbf{s}}$ depending on the steady-state values $\mathbf{X}^{\mathbf{s}}$. Combining these conditions with the original form of the five equations (B.8), (B.11), (B.12), (B.13), and (B.15) can jointly determine $\mathbf{\Lambda^s}$, $\mathbf{\Theta^s}$ and $\mathbf{X^s}$. The above procedure solves the sentiment-driven REE.

It is worth noting that when the volatility of sentiment shock σ_z approaches to zero and the signal s_{jt} precisely reflects the idiosyncratic demand ε_{jt} (i.e., $\lambda \to 1$), the above sentiment-driven REE converges to the fundamental equilibrium described by (29). To see this, with the forecast rule (30), we can write the linearized sentiment-driven equilibrium system as

$$\mathbb{A}^{\mathbf{s}} \begin{bmatrix} \hat{\mathbf{S}}_t^{\mathbf{s}} \\ \hat{\mathbf{X}}_t^{\mathbf{s}} \end{bmatrix} = \mathbb{B}^{\mathbf{s}} \mathbb{E}_t \begin{bmatrix} \hat{\mathbf{S}}_{t+1}^{\mathbf{s}} \\ \hat{\mathbf{X}}_{t+1}^{\mathbf{s}} \end{bmatrix} + \mathbb{C}^{\mathbf{s}} z_t. \tag{B.17}$$

where coefficient matrices are given by

$$\mathbb{A}^{\mathbf{s}} = \begin{bmatrix} \frac{1}{\alpha} & 1 & \frac{1-\alpha}{\alpha} & -1 & 0 & 0 \\ 1 & \alpha & 0 & -1 & 1-\alpha & 0 \\ 0 & -(1-\delta)\frac{K^{\mathbf{s}}}{Y^{\mathbf{s}}} & 0 & -1 & 0 & \frac{C^{\mathbf{s}}}{Y^{\mathbf{s}}} \\ 0 & 0 & 1 & 0 & \nu & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \rho_a & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbb{B}^{\mathbf{s}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{K^{\mathbf{s}}}{Y^{\mathbf{s}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -(\theta-1)\frac{1-\beta(1-\delta)}{1-\alpha+\alpha\theta} & \frac{1-\beta(1-\delta)}{1-\alpha+\alpha\theta} & -(1-\alpha)(\theta-1)\frac{1-\beta(1-\delta)}{1-\alpha+\alpha\theta} & -\frac{1-\beta(1-\delta)}{1-\alpha+\alpha\theta} & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbb{C}^{\mathbf{s}} = \begin{bmatrix} 1-\frac{1-\alpha}{\alpha}\Theta_p^{\mathbf{s}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Define $\begin{bmatrix} \tilde{\mathbf{S}}_t^{\mathbf{s}} \\ \tilde{\mathbf{X}}_t^{\mathbf{s}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{S}}_t^{\mathbf{s}} \\ \hat{\mathbf{X}}_t^{\mathbf{s}} \end{bmatrix} - (\mathbb{A}^{\mathbf{s}})^{-1}\mathbb{C}^{\mathbf{s}} z_t$. Then the dynamic system (B.17) can be transformed as

$$\mathbb{A}^{\mathbf{s}} \begin{bmatrix} \tilde{\mathbf{S}}_t^{\mathbf{s}} \\ \tilde{\mathbf{X}}_t^{\mathbf{s}} \end{bmatrix} = \mathbb{B}^{\mathbf{s}} \mathbb{E}_t \begin{bmatrix} \tilde{\mathbf{S}}_{t+1}^{\mathbf{s}} \\ \tilde{\mathbf{X}}_{t+1}^{\mathbf{s}} \end{bmatrix}. \tag{B.18}$$

By comparing the steady-state conditions of the two equilibria, we find that when $\sigma_z \to 0$ and $\lambda \to 1$, the steady-state values of the aggregate variables in the sentiment-driven equilibrium converge to those in the fundamental equilibrium. By comparing the matrices $\mathbb{A}^{\mathbf{s}}$ and $\mathbb{A}^{\mathbf{f}}$, and $\mathbb{B}^{\mathbf{s}}$ and $\mathbb{B}^{\mathbf{f}}$, we could see that when $\sigma_z \to 0$ and $\lambda \to 1$, $\mathbb{A}^{\mathbf{s}} \to \mathbb{A}^{\mathbf{f}}$ and $\mathbb{B}^{\mathbf{s}} \to \mathbb{B}^{\mathbf{f}}$.

When the fundamental equilibrium has a saddle path around its steady state, there also exists a saddle path around the steady state in the sentiment-driven equilibrium when the sentiment shocks are small. (In this case, given that $\mathbb{A}^{\mathbf{f}}$ is invertible, $\mathbb{A}^{\mathbf{s}}$ must also be invertible.) Then, we could solve the endogenous variables in the vector $\begin{bmatrix} \tilde{\mathbf{S}}_t^{\mathbf{s}} \\ \tilde{\mathbf{X}}_t^{\mathbf{s}} \end{bmatrix}$ with a standard procedure and obtain that $\begin{bmatrix} \tilde{\mathbf{S}}_t^{\mathbf{s}} \\ \tilde{\mathbf{X}}_t^{\mathbf{s}} \end{bmatrix} \approx \begin{bmatrix} \hat{\mathbf{S}}_t^{\mathbf{f}} \\ \hat{\mathbf{X}}_t^{\mathbf{f}} \end{bmatrix}$ for any given state variables a_t and $\hat{k}_{t-1}$. As a result, the policy function of endogenous variables in the sentiment-driven equilibrium can be written as a linear combination of their counterparts in the fundamental equilibrium and the sentiment term z_t, when the volatility of sentiment shocks (σ_z) is small and the signal is precise (λ is close to 1).

Appendix C. The Sentiment-Driven Equilibrium with Information on History

In this section, we solve a sentiment-driven equilibrium with the information structure given by (33). Since agents know the information z_{t-L-1} precisely, they can deduce z_{t-L-1} from the signals in period $t-1$ to $t-L$ so that the information set $\mathbf{\Omega}_{jt}$ becomes

$$\mathbf{\Omega}_{jt} = \left\{ \lambda\varepsilon_{jt} + (1-\lambda)\eta_{0,t}, \mathbf{\Delta}_{1,t}^{L} + \mathbf{v}_{jt}^{L}, z_{t-1-L} \right\} \tag{C.1}$$

where $\mathbf{\Delta}_{1,t}^{L} \equiv [\eta_{1,t}, \eta_{2,t} \cdots, \varepsilon_{zt-L}]'$ where $\eta_{m,t} = \sum_{\tau=m}^{L} \rho_z^{\tau-m} \varepsilon_{zt-\tau}$.

As in Appendix B.2, we base our analysis on a general utility function $U(C_t, N_t) = \log(C_t) - \varphi\frac{N_t^{1+\nu}}{1+\nu}$ and a linearized dynamic system. We still apply the guess-and-verify approach as before.

Intermediate goods producers' forecast rule on the aggregate control variables $\hat{\mathbf{X}}_t^{\mathbf{s}} = [\hat{p}_t, \hat{y}_t, \hat{n}_t, \hat{c}_t]'$ and state variable $\hat{k}_t$ (B.4) now becomes

$$\begin{bmatrix} \hat{\mathbf{X}}_t^{\mathbf{s}} \\ \hat{k}_t^{\mathbf{s}} \end{bmatrix} = \mathbf{\Lambda}^{\mathbf{s}} \hat{\mathbf{S}}_t^{\mathbf{s}} + \mathbf{\Theta}^{\mathbf{s}} \zeta_t^{L} + \mathbf{\Psi}^{\mathbf{s}} z_{t-L-1}. \tag{C.2}$$

where $\mathbf{\Lambda}^{\mathbf{s}}$ are defined as before; $\zeta_t^L = [\varepsilon_{zt}, \varepsilon_{zt-1}, \cdots, \varepsilon_{zt-L}]'$; $\mathbf{\Theta}^{\mathbf{s}} \equiv \begin{bmatrix} \mathbf{\Theta}_p^{\mathbf{s}} \\ \mathbf{\Theta}_y^{\mathbf{s}} \\ \mathbf{\Theta}_n^{\mathbf{s}} \\ \mathbf{\Theta}_c^{\mathbf{s}} \\ \mathbf{\Theta}_k^{\mathbf{s}} \end{bmatrix}$ is now a $5 \times (L+1)$ matrix that collects all the coefficients of all variables before $\{\varepsilon_{zt-\tau}\}_{\tau=0}^{L}$; $\mathbf{\Psi}^{\mathbf{s}} = \left[\mathbf{\Psi}_p^{\mathbf{s}}, \mathbf{\Psi}_y^{\mathbf{s}}, \mathbf{\Psi}_n^{\mathbf{s}}, \mathbf{\Psi}_c^{\mathbf{s}}, \mathbf{\Psi}_k^{\mathbf{s}}\right]'$ is a vector that collects the coefficients of all variables before z_{t-L-1}. We still normalize the first element in $\mathbf{\Theta}_y^{\mathbf{s}}$ to 1, i.e., $\mathbf{\Theta}_{y0}^{\mathbf{s}} = 1$.

Again, we start with deriving $\mathbb{E}\left(\epsilon_{jt}^{\frac{1}{\theta}} P_t Y_t^{\frac{1}{\theta}} | \mathbf{\Omega}_{jt}\right)$ in the optimal decision of Y_{jt}.

$$\begin{aligned} \mathbb{E}\left(\epsilon_{jt}^{\frac{1}{\theta}} P_t Y_t^{\frac{1}{\theta}} | \mathbf{\Omega}_{jt}\right) =& P^{\mathbf{s}} \left(Y^{\mathbf{s}}\right)^{\frac{1}{\theta}} \exp\left[\frac{1}{2}\mathbf{Var}\left(x_{jt}|\mathbf{\Omega}_{jt}\right)\right] \\ & \exp\left[\left(\mathbf{\Lambda}_{pa}^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Lambda}_{ya}^{\mathbf{s}}\right) a_t + \left(\mathbf{\Lambda}_{pk}^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Lambda}_{yk}^{\mathbf{s}}\right) \hat{k}_{t-1}\right] \exp\left[\mathbb{E}\left(x_{jt}|\mathbf{\Omega}_{jt}\right)\right] \end{aligned} \tag{C.3}$$

where $x_{jt} = \frac{1}{\theta}\varepsilon_{jt} + \left(\mathbf{\Theta}_p^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Theta}_y^{\mathbf{s}}\right)\zeta_t^L + \left(\mathbf{\Psi}_p^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Psi}_y^{\mathbf{s}}\right) z_{t-L-1}$ contains all terms involving ε_{jt}, $\{\varepsilon_{zt-\tau}\}_{\tau=0}^{L}$ and

z_{t-L-1}. Then,

$$\mathbb{E}\left(x_{jt}|\mathbf{\Omega}_{jt}\right) = \phi_s\left[\lambda\varepsilon_{jt} + (1-\lambda)\eta_{0,t}\right] + \mathbf{\Phi}\left(\mathbf{\Delta}_{1,t}^L + \mathbf{v}_{jt}^L\right) + \phi_{L+1}z_{t-L-1}, \tag{C.4}$$

where $\mathbf{\Phi} = [\phi_1, \phi_2, \cdots, \phi_L]$ are the coefficients for the information on period $t-1$ to $t-L$; ϕ_s and ϕ_{L+1} are coefficients before s_{jt} and z_{t-L-1}, respectively. Since x_{jt} and all components of $\mathbf{\Omega}_{jt}$ follow a joint normal distribution, then the coefficients ϕ_s, $\mathbf{\Phi}$ and ϕ_{L+1} satisfy

$$[\phi_s \ \ \mathbf{\Phi} \ \ \phi_{L+1}] = \mathbf{\Sigma}_{x,\mathbf{\Omega}}\mathbf{\Sigma}_{\mathbf{\Omega},\mathbf{\Omega}}^{-1}, \tag{C.5}$$

where $\mathbf{\Sigma}_{x,\mathbf{\Omega}}$ is a row vector which consists of the covariance between x_{jt} and each element in $\mathbf{\Omega}_{jt}$; and $\Sigma_{\mathbf{\Omega},\mathbf{\Omega}}$ is the variance-covariance matrix of $\mathbf{\Omega}_{jt}$.

Substituting the above expression into (10), we express the best response of y_{jt} as

$$\begin{aligned} y_{jt} \quad = \quad & \bar{y} + \frac{1}{(1-\alpha)\,\varrho}(a_t + \alpha k_{jt-1}) + \frac{1}{\varrho}\left(\mathbf{\Lambda}_{pa}^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Lambda}_{ya}^{\mathbf{s}}\right)a_t + \frac{1}{\varrho}\left(\mathbf{\Lambda}_{pk}^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Lambda}_{yk}^{\mathbf{s}}\right)\hat{k}_{t-1} \\ & + \frac{1}{\varrho}\phi_s\left[\lambda\varepsilon_{jt} + (1-\lambda)\eta_{0,t}\right] + \frac{1}{\varrho}\mathbf{\Phi}\left(\mathbf{\Delta}_{1,t}^L + \mathbf{v}_{jt}^L\right) + \frac{1}{\varrho}\phi_{L+1}z_{t-L-1}. \end{aligned} \tag{C.6}$$

Aggregating $Y_{jt} = \exp(y_{jt})$ gives aggregate output Y_t. We further log-linearize this equation around the steady state and obtain

$$\begin{aligned} \hat{y}_t \quad = \quad & \frac{1}{\varrho}\left(\frac{1}{1-\alpha}\mathbf{\Lambda}_{pa}^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Lambda}_{ya}^{\mathbf{s}}\right)a_t + \frac{1}{\varrho}\left(\frac{\alpha}{1-\alpha} + \mathbf{\Lambda}_{pk}^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Lambda}_{yk}^{\mathbf{s}}\right)\hat{k}_{t-1} \\ & + \frac{1}{\varrho}\phi_s(1-\lambda)\eta_{0,t} + \frac{1}{\varrho}\mathbf{\Phi}\mathbf{\Delta}_{1,t}^L + \frac{1}{\varrho}\phi_{L+1}z_{t-L-1}. \end{aligned} \tag{C.7}$$

Substituting (C.6) into production function (7) yields N_{jt} and aggregating N_{jt} gives aggregate labor N_t. Then we log-linearize the resulted equation around the steady state and obtain

$$\begin{aligned} \hat{n}_t \quad = \quad & \frac{1}{(1-\alpha)\varrho}\left(\frac{\theta-1}{\theta} + \mathbf{\Lambda}_{pa}^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Lambda}_{ya}^{\mathbf{s}}\right)a_t + \frac{1}{(1-\alpha)\varrho}\left(\alpha\frac{\theta-1}{\theta} + \mathbf{\Lambda}_{pk}^{\mathbf{s}} + \frac{1}{\theta}\mathbf{\Lambda}_{yk}^{\mathbf{s}}\right)\hat{k}_{t-1} \\ & + \frac{1}{(1-\alpha)\varrho}\phi_s(1-\lambda)\eta_{0,t} + \frac{1}{(1-\alpha)\varrho}\mathbf{\Phi}\mathbf{\Delta}_{1,t}^L + \frac{1}{(1-\alpha)\varrho}\phi_{L+1}z_{t-L-1}. \end{aligned} \tag{C.8}$$

The labor supply curve (B.12) and resource constraint (B.13) are as before. The Euler equation of investment in this case becomes

$$1 = \beta\mathbb{E}_t\frac{C_t}{C_{t+1}}\left\{\begin{array}{c} 1-\delta+\Gamma_k A_{t+1}^{\frac{\theta-1}{1-\alpha+\alpha\theta}}K_t^{-\frac{1}{1-\alpha+\alpha\theta}} \times \exp\left[\frac{\mathbf{\Lambda}_{ya}^{\mathbf{s}}+(\theta-1)(1-\alpha)\mathbf{\Lambda}_{pa}^{\mathbf{s}}}{1-\alpha+\alpha\theta}a_{t+1} + \frac{\mathbf{\Lambda}_{yk}^{\mathbf{s}}+(\theta-1)(1-\alpha)\mathbf{\Lambda}_{pk}^{\mathbf{s}}}{1-\alpha+\alpha\theta}\hat{k}_t\right] \\ \times\left[\Gamma_{k1}\exp\left[\frac{1}{\theta}\mathbf{\Theta}_y^{\mathbf{s}}\zeta_{t+1}^L + \frac{\theta-1}{\theta}\frac{\phi_s}{\varrho}(1-\lambda)\eta_{0,t+1} + \frac{\theta-1}{\theta}\frac{1}{\varrho}\mathbf{\Phi}\mathbf{\Delta}_{1,t+1}^L + \left(\frac{1}{\theta}\mathbf{\Psi}_y^{\mathbf{s}} + \frac{\theta-1}{\theta}\frac{1}{\varrho}\phi_{L+1}\right)z_{t-L}\right]\right. \\ \left.-\Gamma_{k2}\exp\left[-\mathbf{\Theta}_p^{\mathbf{s}}\zeta_{t+1}^L + \frac{1}{1-\alpha}\frac{\phi_s}{\varrho}(1-\lambda)\eta_{0,t+1} + \frac{1}{1-\alpha}\frac{1}{\varrho}\mathbf{\Phi}\mathbf{\Delta}_{1,t+1}^L + \left(\frac{1}{1-\alpha}\frac{1}{\varrho}\phi_{L+1} - \mathbf{\Psi}_p^{\mathbf{s}}\right)z_{t-L}\right]\right] \end{array}\right\},$$

where Γ_k, Γ_{k1} and Γ_{k2} are constants depending on parameters and the steady state. We further log-linearize the above equation and obtain

$$
\begin{aligned}
0 \quad = \quad & \hat{c}_t - \mathbb{E}_t\hat{c}_{t+1} + \frac{1-\beta(1-\delta)}{1-\alpha+\alpha\theta}\Bigg\{ \left[(\theta-1) + \mathbf{\Lambda}^{\mathbf{s}}_{ya} + (\theta-1)(1-\alpha)\,\mathbf{\Lambda}^{\mathbf{s}}_{pa}\right]\mathbb{E}_t a_{t+1} \\
& + \left[\mathbf{\Lambda}^{\mathbf{s}}_{yk} + (\theta-1)(1-\alpha)\,\mathbf{\Lambda}^{\mathbf{s}}_{pk} - 1\right]\hat{k}_t + \frac{(1-\alpha+\alpha\theta)\Gamma_{k1}}{\Gamma_{k1}-\Gamma_{k2}}\left[\mathbb{E}_t\left(\frac{1}{\theta}\mathbf{\Theta}^{\mathbf{s}}_y\zeta^L_{t+1}\right) + \frac{\theta-1}{\theta}\frac{\phi_s}{\varrho}(1-\lambda)\mathbb{E}_t\eta_{0,t+1}\right. \\
& \left. + \frac{\theta-1}{\theta}\frac{1}{\varrho}\mathbf{\Phi}\mathbf{\Delta}^L_{1,t+1} + \left(\frac{1}{\theta}\mathbf{\Psi}^{\mathbf{s}}_y + \frac{\theta-1}{\theta}\frac{1}{\varrho}\phi_{L+1}\right)z_{t-L}\right] - \frac{(1-\alpha+\alpha\theta)\Gamma_{k2}}{\Gamma_{k1}-\Gamma_{k2}}\left[-\mathbb{E}_t\left(\mathbf{\Theta}^{\mathbf{s}}_p\zeta^L_{t+1}\right) + \right. \\
& \left.\left. \frac{1}{1-\alpha}\frac{\phi_s}{\varrho}(1-\lambda)\mathbb{E}_t\eta_{0,t+1} + \frac{1}{1-\alpha}\frac{1}{\varrho}\mathbf{\Phi}\mathbf{\Delta}^L_{1,t+1} + \left(\frac{1}{1-\alpha}\frac{1}{\varrho}\phi_{L+1} - \mathbf{\Psi}^{\mathbf{s}}_p\right)z_{t-L}\right]\right\}.
\end{aligned} \tag{C.9}
$$

Then we have the linearized dynamic system (C.7), (C.8), (C.9), (B.12) and (B.13) for $\left\{\hat{p}_t, \hat{y}_t, \hat{n}_t, \hat{c}_t, \hat{k}_t\right\}$ in the sentiment-driven equilibrium. The original form of these five equations pin down the steady-state values of these variables $\mathbf{X}^{\mathbf{s}} = [C^{\mathbf{s}}, P^{\mathbf{s}}, Y^{\mathbf{s}}, N^{\mathbf{s}}, K^{\mathbf{s}}]'$.

Matching coefficients in the conjecture rule (C.2) and the above dynamic system determines the coefficient matrices $\mathbf{\Lambda}^{\mathbf{s}}$, $\mathbf{\Theta}^{\mathbf{s}}$ and $\mathbf{\Psi}^{\mathbf{s}}$ which also depend on $\mathbf{X}^{\mathbf{s}}$. Combining these conditions with the original form of equations (C.7), (C.8), (C.9), (B.12) and (B.13) can jointly determine $\mathbf{\Lambda}^{\mathbf{s}}$, $\mathbf{\Theta}^{\mathbf{s}}$, $\mathbf{\Psi}^{\mathbf{s}}$ and $\mathbf{X}^{\mathbf{s}}$. The above procedure solves the sentiment-driven REE with information on history.

In addition, we can show that $\mathbf{\Psi}^{\mathbf{s}}$ are all zero. To see that, we notice that $\Sigma_{\mathbf{\Omega},\mathbf{\Omega}}$ is a $(L+2)\times(L+2)$ matrix satisfying

$$
\mathbf{\Sigma}_{\mathbf{\Omega},\mathbf{\Omega}} = \begin{bmatrix} \widetilde{\mathbf{\Sigma}}_{\mathbf{\Omega},\mathbf{\Omega}} & 0 \\ 0 & \mathrm{var}\left(z_{t-L-1}\right) \end{bmatrix},
$$

where $\widetilde{\mathbf{\Sigma}}_{\mathbf{\Omega},\mathbf{\Omega}}$ is the variance-covariance matrix of the first $L+1$ elements in the information set $\mathbf{\Omega}_{jt}$ which are all free of the term z_{t-L-1}. Note that all the first $L+1$ elements are correlated with each other, then $\widetilde{\mathbf{\Sigma}}_{\mathbf{\Omega},\mathbf{\Omega}}$ is a non-zero matrix. Using the stochastic process of z_t, it is easy to verify that $\widetilde{\mathbf{\Sigma}}_{\mathbf{\Omega},\mathbf{\Omega}}$ is invertible. Thus we have

$$
\mathbf{\Sigma}^{-1}_{\mathbf{\Omega},\mathbf{\Omega}} = \begin{bmatrix} \widetilde{\mathbf{\Sigma}}^{-1}_{\mathbf{\Omega},\mathbf{\Omega}} & 0 \\ 0 & 1/\mathrm{var}\left(z_{t-L-1}\right) \end{bmatrix}.
$$

Since $\mathbf{\Sigma}_{x,\mathbf{\Omega}} = \begin{bmatrix} \widetilde{\mathbf{\Sigma}}_{x,\mathbf{\Omega}}, & 0 \end{bmatrix}$ is a $1\times(L+2)$ row vector where the last entry is zero, then the last element in $\mathbf{\Sigma}_{x,\mathbf{\Omega}}\mathbf{\Sigma}^{-1}_{\mathbf{\Omega},\mathbf{\Omega}}$ must be 0, i.e., $\phi_{L+1} = 0$. Then by equations (C.7) and (C.8), we have $\mathbf{\Psi}^{\mathbf{s}}_y = \frac{1}{\varrho}\phi_{L+1} = 0$ and $\mathbf{\Psi}^{\mathbf{s}}_n = \frac{1}{(1-\alpha)\varrho}\phi_{L+1} = 0$. Equation (C.9) implies that $\mathbf{\Psi}^{\mathbf{s}}_p = 0$. Equation (B.12) implies that $\mathbf{\Psi}^{\mathbf{s}}_c = 0$. Equation (B.13) implies that $\mathbf{\Psi}^{\mathbf{s}}_k = 0$. That is, $\mathbf{\Psi}^{\mathbf{s}}$ are all zero and z_{t-L-1} does not affect the macroeconomy, which is consistent with the rationale that precise information on the past sentiments eliminate persistent impacts of sentiments in Section 3.3.

It is also worth noting that $\widetilde{\mathbf{\Sigma}}_{\mathbf{x},\mathbf{\Omega}}$ and $\widetilde{\mathbf{\Sigma}}^{-1}_{\mathbf{\Omega},\mathbf{\Omega}}$ are all non zeros, thus $\mathbf{\Phi} = \widetilde{\mathbf{\Sigma}}_{\mathbf{x},\mathbf{\Omega}}\widetilde{\mathbf{\Sigma}}^{-1}_{\mathbf{\Omega},\mathbf{\Omega}}$ is a non-zero vector, which implies that the endogenous aggregate variables $\hat{y}_t$, $\hat{n}_t$,$\hat{p}_t$, $\hat{c}_t$ and $\hat{k}_t$ also depend on past sentiments $\{\varepsilon_{zt-\tau}\}^L_{\tau=1}$, or equivalently, sentiment shocks z_t can generate persistent effects.

Appendix D. The Sentiment-Driven Equilibrium with Information on History of Endogenous Variables

In this section, we allow agents to have noisy information on the past realization of aggregate endogenous variables instead of the sentiment shocks. For simplicity, we only allow them to have information on the price level in the past. Specifically, firms know $\{\hat{p}_{t-\tau}\}_{\tau=1}^{L}$ with noises $\{v_{jt-\tau}\}_{\tau=1}^{L}$ and know the full information before $t-L$ precisely. Therefore, the information set $\mathbf{\Omega}_{jt}$ becomes

$$\mathbf{\Omega}_{jt} = \left\{ \lambda \varepsilon_{jt} + (1-\lambda) \sum_{\tau=0}^{L} \rho_z^{\tau} \varepsilon_{zt-\tau}, \mathbf{p}_{t-1}^{L} + \mathbf{v}_{jt-1}^{L}, z_{t-L-1} \right\}, \tag{D.1}$$

where $\mathbf{p}_{t-1}^{L} \equiv [\hat{p}_{t-1}, \hat{p}_{t-2}, \cdots, \hat{p}_{t-L}]'$ and $\mathbf{v}_{jt-1}^{L} \equiv [v_{jt-1}, v_{jt-2}, \cdots, v_{jt-L}]'$.

Intermediate goods producers' forecast rule on the aggregate control variables $\hat{\mathbf{X}}_t^{\mathbf{s}} = [\hat{p}_t, \hat{y}_t, \hat{n}_t, \hat{c}_t]'$ and state variable $\hat{k}_t$ is still given by (C.2). To facilitate the presentation, in this section we normalize the first element of $\mathbf{\Theta}_p^{\mathbf{s}}$ to 1 instead of $\mathbf{\Theta}_{y0}^{\mathbf{s}} = 1$. This re-normalization does not alter the equilibrium solved in Appendix C essentially.

Denote $\zeta_t^L = \begin{bmatrix} \zeta_{(0,m)t}^L \\ \zeta_{(m+1,L)t}^L \end{bmatrix}$ where $\zeta_{(m,n)t}^L$ is the vector of the $(m+1)$-th to $(n+1)$-th element of ζ_t^L. Also denote $\mathbf{\Theta}_p^{\mathbf{s}} = [\mathbf{\Theta}_{p(0,m)}^{\mathbf{s}} \quad \mathbf{\Theta}_{p(m+1,L)}^{\mathbf{s}}]$ where $\mathbf{\Theta}_{p(m,n)}^{\mathbf{s}}$ is the vector of coefficients of the price level $\hat{p}_t$ before the elements of $\zeta_{(m,n)t}^L$. Then we could express the forecast rule of the price level as

$$\hat{p}_t = \mathbf{\Lambda}_p^{\mathbf{s}} \hat{\mathbf{S}}_t^{\mathbf{s}} + \begin{bmatrix} \mathbf{\Theta}_{p(0,m)}^{\mathbf{s}} & \mathbf{\Theta}_{p(m+1,L)}^{\mathbf{s}} \end{bmatrix} \begin{bmatrix} \zeta_{(0,m)t}^L \\ \zeta_{(m+1,L)t}^L \end{bmatrix} + \mathbf{\Psi}_p^{\mathbf{s}} z_{t-L-1}. \tag{D.2}$$

We first consider the case that $m = 0$. Since the firms know all the information on aggregate fundamentals about a_t and $\hat{k}_{t-1}$ and periods before $t-L$, then they know all of the terms in $\left(\mathbf{\Lambda}_p^{\mathbf{s}} \hat{\mathbf{S}}_{t-L}^{\mathbf{s}} + \mathbf{\Theta}_{p(1,L)}^{\mathbf{s}} \zeta_{(1,L)t-L}^L + \mathbf{\Psi}_p^{\mathbf{s}} z_{t-2L-1}\right)$. Then they could infer

$$\begin{aligned} & \varepsilon_{zt-L} + v_{jt-L} \\ = \ & \mathbf{\Theta}_{p(0,0)}^{\mathbf{s}} \varepsilon_{zt-L} + v_{jt-L} \\ = \ & (\hat{p}_{t-L} + v_{jt-L}) - \left(\mathbf{\Lambda}_p^{\mathbf{s}} \hat{\mathbf{S}}_{t-L}^{\mathbf{s}} + \mathbf{\Theta}_{p(1,L)}^{\mathbf{s}} \zeta_{(1,L)t-L}^L + \mathbf{\Psi}_p^{\mathbf{s}} z_{t-2L-1} \right), \end{aligned} \tag{D.3}$$

where the first equality is due to $\mathbf{\Theta}_{p(0,0)}^{\mathbf{s}} = 1$. This equation tells us that firms know ε_{zt-L} with the noise v_{jt-L} by separating the information $\left(\mathbf{\Lambda}_p^{\mathbf{s}} \hat{\mathbf{S}}_{t-L}^{\mathbf{s}} + \mathbf{\Theta}_{p(1,L)}^{\mathbf{s}} \zeta_{(1,L)t-L}^L + \mathbf{\Psi}_p^{\mathbf{s}} z_{t-2L-1}\right)$ from the noisy price level $\hat{p}_{t-L} + v_{jt-L}$.

We next consider the case that $m = 1$. Similarly, firms could infer

$$\begin{aligned} & \varepsilon_{zt-L+1} + \tilde{v}_{jt-L+1} \\ = \quad & \mathbf{\Theta}^{\mathbf{s}}_{p(0,0)} \varepsilon_{zt-L+1} + v_{jt-L+1} - \mathbf{\Theta}^{\mathbf{s}}_{p(1,1)} v_{jt-L} \\ = \quad & (\hat{p}_{t-L+1} + v_{jt-L+1}) - \left[\mathbf{\Lambda}^{\mathbf{s}}_{p} \hat{\mathbf{S}}^{\mathbf{s}}_{t-L+1} + \mathbf{\Theta}^{\mathbf{s}}_{p(1,1)} \left(\varepsilon_{zt-L} + v_{jt-L} \right) + \mathbf{\Theta}^{\mathbf{s}}_{p(2,L)} \mathbf{\Delta}^{L+1}_{(2,L)t-L+1} + \mathbf{\Psi}^{\mathbf{s}}_{p} z_{t-2L} \right]. \end{aligned} \tag{D.4}$$

That is, once obtaining $\varepsilon_{zt-L} + v_{jt-L}$ from (D.3), along with the information on the aggregate fundamentals $\hat{\mathbf{S}}^{\mathbf{s}}_{t-L+1}$[7] and all the precise information before $t - L$, $\left(\mathbf{\Theta}^{\mathbf{s}}_{p(2,L)} \mathbf{\Delta}^{L+1}_{(2,L)t-L+1} + \mathbf{\Psi}^{\mathbf{s}}_{p} z_{t-2L}\right)$, the firms could know ε_{zt-L+1} with the noise $\tilde{v}_{jt-L+1}$.

Here, the noise term is redefined as $\tilde{v}_{jt-L+1} = v_{jt-L+1} - \mathbf{\Theta}^{\mathbf{s}}_{p(1,1)} v_{jt-L}$ which is also normally distributed, given v_{jt-L+1} and v_{jt-L} both normally distributed. In this way, using the information set (D.1) and forecast rule (D.2), firms could infer all elements of $[\varepsilon_{zt}, \varepsilon_{zt-1}, \cdots \varepsilon_{zt-L}]'$ with noises $\widetilde{\mathbf{v}}^{L}_{jt} = \{\tilde{v}_{jt-\tau}\}^{L}_{\tau=1}$ iteratively. Letting $\overline{\mathbf{v}}^{L}_{jt-1} = \left\{ \sum_{\tau=m}^{L} \rho_z^{\tau-m} \tilde{v}_{jt-\tau} \right\}^{L}_{m=1}$, the information set could be rewritten as

$$\mathbf{\Omega}_{jt} = \left\{ \lambda \varepsilon_{jt} + (1-\lambda) \sum_{\tau=0}^{L} \rho_z^{\tau} \varepsilon_{zt-\tau}, \mathbf{Z}^{L}_{t-1} + \overline{\mathbf{v}}^{L}_{jt-1}, z_{t-L-1} \right\}, \tag{D.5}$$

which is the essentially the same as (33). Given the existence of the sentiment-driven equilibrium in the previous section, the existence of the sentiment-driven equilibrium here is also established. We could solve the sentiment-driven equilibrium with the same procedure described in the previous section.

[7] To prevent information revelation via the state variables (i.e., K_{t-1}), we assume that the aggregate state variables also contain noises in this case.