A Tale of Two Dispersions: Wage and Firm Size*

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Abstract

The traditional literature treats wage dispersion and firm dynamics, which are closely connected to each other, in isolation. This paper delivers a unified treatment to wage dispersion and firm-size distribution by developing a real-option approach. The model is tractable with analytical solution, generating the following testable implications. Firstly, the distribution of firm size is uni-modal, right-skewed with a Paretian tail, which is in line with the empirical findings, in particular the Zipf Law. So is that of wage dispersion. Secondly, the incumbents prefer to preserve the pattern of labor hoarding rather than exiting the market when hit by (not too severely) negative productivity shock. Thirdly, in addition to the effect in standard search and matching theory, the labor market tightness is also found to produce additional transition mechanisms to the unemployment rate. Fourthly, the model predicts that, the larger the firm is, the longer the firm will survive at the market.

Key Words: Endogenous Job Destruction, Wage Dispersion, Firm Dynamics

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1 Introduction

This paper delivers a *unified* treatment to wage dispersion and firm-size distribution by developing a real-option approach. The model is tractable with analytical solution, generating the following testable implications. Firstly, the distribution of firm size is uni-modal, right-skewed with a Paretian tail, which is in line with the empirical findings, in particular the Zipf Law. So is that of wage dispersion. Secondly, as in Mortensen and Pissarides (1994), the incumbents prefer to preserve the pattern of labor hoarding rather than exiting the market when hit by (not too severely) negative productivity shock. Thirdly, in addition to the effect in standard search and matching theory, the labor market tightness is also found to produce additional transition mechanisms to the unemployment rate. Fourthly, the model predicts that, the larger the firm is, the longer the firm will survive in the market.

This paper is mainly related to three strands of literature.

The first strand is about wage dispersion. Inspired by the robust empirical findings of different pay to nearly identical workers, the seminal research by Burdett and Mortensen (1998) employs on-the-job-search to produce a wage distribution of the ex ante identical workers. Mortensen (2000) offers an excellent survey of the research progress in this direction. However, as Mortensen (2000) points out, although the Burdett-Mortensen model has a logically consistent style, the predicted distribution of the model is far from the wage dispersion in the real life. As Moscarini (2005) and Jolivet, Postel-Vinay, and Robin (2006), among others, show that the wage dispersion of developed countries roughly follows a uni-modal, Paretian right tail distribution.

The second one is about firm dynamics. The formal discussion on firm dynamics can be dated back at least to the works by Gibrat (Sutton, 1997) and Zipf (Cabral and Mata, 2003), who proposed the distribution of firm size. Based on the works by Simon and Bonini (1958), Lucas Jr (1978) came up with a micro-foundation for the distribution of firm size. Klette and Kortum (2004) use the approach of R&D in economic growth to verify the distribution while Luttmer (2007, 2011, 2010) consider Brownian-motion models to fit the distribution. Additionally, Gabaix (1999, 2009), among others, also relys on Brownian motion to pin down the Paretian distribution of firm size.

There are mainly three kinds of paradigm for the research on industry and firm dynamics. First of all, Ericson and Pakes (1995) comes up with models with a finite number of firms by characterizing the dynamics of a limited number of heterogeneous firms which compete in same industry. Basically they adopt the concept of *Markov Perfect Equilibrium (MPE)*. Secondly, Hopenhayn (1992) initiates the models with a continuum of firms and employs the concept of *Stationary Equilibrium (SE)*. Finally, Weintraub, Benkard, and Van Roy (2011) invents the *Oblivious Equilibrium (OE)*, attempting to enjoy the simplifying benefits of models with infinite number of firms but preserve the more realistic setting from a finite model. In this paper, I restrict my attention to the second paradigm, using the notion of

Stationary Equilibrium (SE), in which the time subscript is suppressed for simplicity.

As indicated above, both wage dispersion and firm dynamics have been extensively documented in the literature. However, it is surprising that, to my best knowledge, nearly all of the research treat the wage dispersion and firm dynamics in isolation. One of the rare exceptions is Mortensen and Pissarides (1994), which considers the endogenous job destruction in two-sided matching model. The main drawback is that their model fails to generate the distribution pattern in real data. Instead, traditional literature on firm and industry dynamics usually assumes away the unemployment issue associating with firms' exiting, mostly due to the concern of tractability, and thus completely ignores the churning cost, such as the loss due to unemployment. However, the production involves both the firms and the workers. Moreover, the welfare of the workers is closely related to that of the firms in their life-cycle of entry, recruitment, and development and exiting. One of the key contributions of this paper is to deliver a tractable model to consider the distributions of both wage and firm size with one shot.

The third one is about unemployment. The discussion of unemployment has been a classic issue in macroeconomics, which has been explored not only in short run (Andolfatto, 1996), but also in long run (Berentsen, Menzio, and Wright, 2011). Typically, the unemployment is best discussed with the framework of search and matching (Mortensen and Pissarides, 1994, 1999a,b; Pissarides, 2000). However, there is only one transmission mechanism in the classic search-and-matching model on unemployment. Another contribution of this paper is to propose and identify additional transmission mechanism on unemployment, which is explained in the details in Section 3.

Finally, the paper employs the technique called real option in the context of Brownian motion. Dixit, Dixit, and Pindyck (1994), Chang (2004) and Stokey (2009), among others, offer excellent survey of the application of real option in economics and finance. The key merit of using real option is due to its high tractability.

The rest of the paper proceeds as follows. Section 2 sets up the baseline model, establishing the interaction between firms and workers and characterizing the entry and exit decisions by the firms. Section 3 uses a general-equilibrium analysis to close the model, offering the distribution of firm size as well as that of wage dispersion. Besides, we characterize the novel transmission mechanism on unemployment rate in this section. Section 4 lists some potential extension and Section 5 concludes. The proof omitted in the text is documented in the Appendix.

2 Basic Model

2.1 Environment

Time is continuous with two kinds of agents, workers and firms. Each worker is endowed with one unit of indivisible labor supply, $i.e.$, $l \equiv 1$. The status of the workers is categorized into two states: employed and unemployed. Firms, on the other hand, are possible to be idle, incumbent, exit or potentially wait to entry. Without loss of generality, the total number of workers is normalized to be $L \equiv 1$ and thus $u \in (0, 1)$ not only stands for the number or measure of the unemployed, but also for the unemployment rate. As in the standard search and matching theory, I introduce into the economy with an aggregate random matching function $m(u, v)$, which is assumed to be homogeneous of degree one.

We denote the labor market tightness θ and probability rate $q(\theta)$ as $\theta \equiv \frac{v}{u}$ and $q(\theta) \equiv \frac{m(u,v)}{v}$ respectively. Therefore $q(\theta)$ is the probability rate for any idle firm to meet a worker while $\theta q(\theta)$ is that for any unemployed worker to meet a potential startup.

Once both sides are matched and they reach mutual agreement to production, then the firm's flow of gross revenue is $R = Py = Ppf(l)$, where P denotes the market price while $y = pf(l)$ is the firm's production function with her idiosyncratic productivity p and the worker's labor input l. In this benchmark model, I assume away the role of capital in the production and assume that $f(l) = l$. To discuss the firm-level dynamics, I assume throughout this Section 2 that the market price P is exogenously given. Since I restrict my attention to stationary distribution, without loss of generality, the market price can be normalized to be $P \equiv 1$.

Throughout the paper I assume that there is no on-the-job search and all kinds of information are publicly observable and verifiable.

2.2 Firms

The incumbent *firms' productivity* p, which is also often treated as an indicator for *firm size* in the literature, is assumed to follow Geometric Brownian Motion (GBM), $i.e.$,

$$dp = \mu_p p dt + \sigma_p p dZ$$

where the constants μ_p and σ_p are non-negative and Z is a standard Wiener Process.[1] Note that since the evolution of each incumbent's productivity is assumed to independent to each other, I suppress

[1] He (2008) has an argument to validate the GBM assumption for the evolution of firm size.

the identity subscript i for any incumbent i.

Besides, I assume that there is an exogenous shock such that the incumbents are eliminated out of their business with a Poisson death rate δ. The introduction of the Poisson rate is mainly motivated by the following three factors. First of all, Since GBM is a non-stationary process, in the absence of Poisson death rare, it does not converge to a well-defined distribution in the long run. Secondly, the Poisson death rate can be treated as the extremely negative shock to productivity, revenue or profts. Finally, the imposition of Possion rate can be referred as the retirement or promotion of the workers.

For technical reasons, we assume the Poisson death rate δ is large enough. In particular, we assume that

Assumption 2.1. *The Poisson death rate δ is high enough:* $\delta > \max\{\mu_p, \sigma_p^2 - \mu_p\}$.

In the benchmark, we assume away all the other sources of operation costs, taking corporate tax and managerial costs for example, for the incumbents.

For potential entrants, they can have an access to drawing their initial productivity p whose distribution is assumed to follow the uniform distribution $G(p)$ with $p \in [\overline{p} - \sigma_{\overline{p}}, \overline{p} + \sigma_{\overline{p}}]$, provided that they can credibly commit to paying the irreversible entry cost $c > 0$. Once meeting with a worker and both sides agree to some labor contract, the startup will then begin her operation with her drawed productivity p. If the potential entrant does meet any worker after paying the cost flow c and drawing her productivity p, then the production will be obsolete and she would have to pay another cost flow if she wants to participate in the matching game in the labor market.

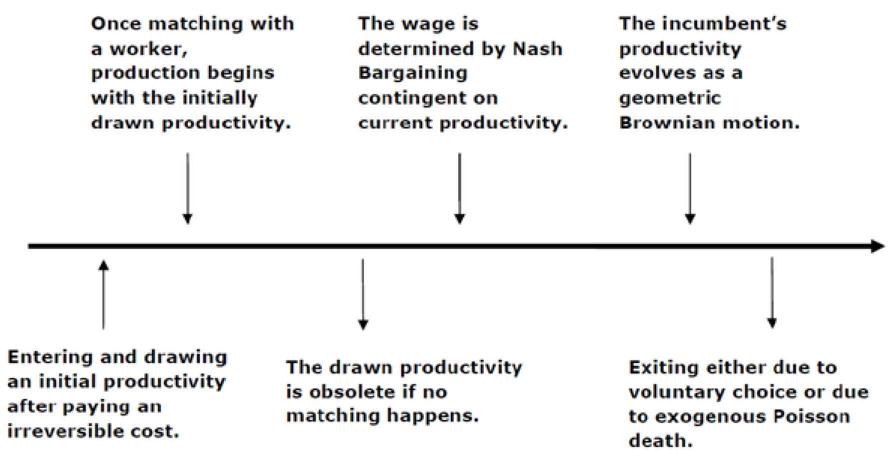


Figure 1: Timeline of firm's decision

2.3 Workers

The wage flow is determined by a Nash bargaining in which each party gains a constant share of the total surplus from matching $S(p)$ at each time point, as long as the firm is not disbanned or exogenously removed out of the market by some unforeseeable shock. Therefore, the firms and the workers always agree with each other about whether and when to terminate the labor contract. Consequently, it is neither possible nor necessary to distinguish between layoff and quit.

Once the workers are unemployed, they can get the flow of unemployment pension $\varrho \geq 0$ from the government until they get a new job. Besides, the workers are assumed to suffer the monetary form of psychic costs as well as searching costs $\kappa > 0$. For technical reasons, we assume that κ is larger than ϱ. Denote the gross suffering in the monetary cost as

$$s \equiv \kappa - \varrho > 0$$

In the benchmark model of this section, we put aside the problem of *self-financing*, in which the unemployment pension has to be covered by the tax imposed on the operating firms.

Then the function for the unemployed can be written as follows.

$$rU = -s + \theta q(\theta) \int_{\max\{p_e, \overline{p} - \sigma_{\overline{p}}\}}^{\overline{p} + \sigma_{\overline{p}}} \beta S(p) dG(p)$$

where p_e is reservation productivity level only above which will the firm and the worker agree to sign a labor contract and engage in production, $S(p)$ is the flow of total surplus from matching and $\beta \in (0, 1)$ is the bargaining power of workers.

For simplicity, we assume that the lower bound p_L is sufficiently large that we always have $p_L > p_e$ and thus the above equation can be rewritten as below.

$$rU = -s + \theta q(\theta) \int_{\overline{p} - \sigma_{\overline{p}}}^{\overline{p} + \sigma_{\overline{p}}} \beta S(p) dG(p)$$

It will turn out the cut-off point for the firm's exiting from the market p_e is endogenous and thus the assumption that $p_L \equiv \overline{p} - \sigma_{\overline{p}} > p_e$ should be checked in details. We will address the *verification* problem later.

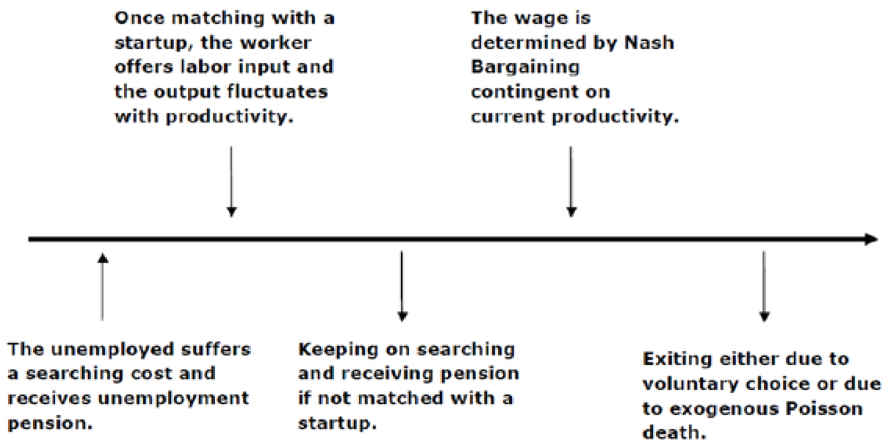


Figure 2: Timeline of worker's decision

2.4 Wages

The wage flow is determined by a Nash bargaining in which each party gains a constant share of the total surplus from matching $S(p)$ at each time point, as long as the firm is not disbanned or exogenously removed out of the market by some unforeseeable shock. Therefore, the firms and the workers always agree with each other about whether and when to terminate the labor contract. Consequently, it is neither possible nor necessary to distinguish between layoff and quit.

By the rule of Nash bargaining, the wage flow $w(p)$ should always satisfy the following restriction.

$$\begin{aligned} p - w(p) &= (1-\beta)[(p-w) + (w - rU)] \\ &= (1-\beta)(p - rU) \end{aligned}$$

Thus we have

$$w(p) = \beta p + (1-\beta) rU$$

In contrary to the fact that the labor force is normalized to be one, there are free entry and exit for the firm side. The free entry condition requires the following restrictions in equilibrium.

$$c = q(\theta) \int_{\overline{p}-\sigma_{\overline{p}}}^{\overline{p}+\sigma_{\overline{p}}} (1-\beta) S(p) dG(p)$$

Thus we have

$$rU = -s + \frac{\beta c \theta}{1-\beta}$$

and the wage flow can be rewritten as below.

$$w(p) = \beta(p + c\theta) - (1-\beta)s$$

It then remains to figure out the steay-state market tightness θ to pin down the wage rate.

3 Equilibrium

To characterize the optimal exit productivity p_e and the labor market tightness θ, we first lay down the following lemma and proposition.

Lemma 1. ***(Feynman-Kac)*** *The evolution and the boundary conditions of the total surplus $S(p)$ from a matching with current productivity p are characterized by the following equations.*

$$rS(p) = p - rU - \delta S(p) + S'(p)\mu_p p + \frac{1}{2} S''(p)\sigma_p^2 p^2$$
$$S(p_e) = 0 \quad \textit{(Value Matching)}$$
$$S'(p_e) = 0 \quad \textit{(Smooth Pasting)}$$

Proposition 1. *The total surplus $S(p)$ from a matching with current productivity p and the optimal exit productivity p_e is captured by the following equations.*

1. ***(Total surplus** $S(p)$**)***

$$S(p) = \Xi(p) - \Xi(p_e)(\frac{p}{p_e})^{\alpha}$$

where

$$\Xi(p) \equiv ap - b$$

and

$$a \equiv \frac{1}{r+\delta-\mu_p} > 0$$
$$b \equiv \frac{rU}{r+\delta} = \frac{1}{r+\delta}(-s+\frac{\beta c\theta}{1-\beta}) > 0$$

and α is the negative root of the following equation

$$\frac{\sigma_p^2}{2}\alpha(\alpha-1) = r+\delta-\alpha\mu_p$$

2. ***(The optimal exit productivity** p_e**)***

$$\begin{aligned} p_e &= (\frac{\alpha}{\alpha-1})(\frac{r+\delta-\mu_p}{r+\delta})rU \\ &= (\frac{\alpha}{\alpha-1})(\frac{r+\delta-\mu_p}{r+\delta})(-s+\frac{\beta c\theta}{1-\beta}) \\ &= (\frac{\alpha}{\alpha-1})(\frac{b}{a}) \end{aligned}$$

Corollary 1. ***(Time Consistency)*** *The optimal exiting rule p_e is independent of firm's current productivity p.*

Corollary 2. ***(The Option to Wait before Exit)*** *At the optimal exiting productivity p_e, we have*

$$\Xi(p_e) = \frac{b}{\alpha-1} < 0$$

To make sure that $p_e \geq 0$ is always satisfied, we assume that $\theta \geq \widetilde{\theta}$, where $\widetilde{\theta}$ is defined as

$$-s+\frac{\beta c\widetilde{\theta}}{1-\beta} = 0$$

Denote $T_{p_e,\widetilde{p}}(p) = \inf\{t : p_t \notin (p_e, \widetilde{p})\}$. Then the expected hitting time is given by

$$T_{p_e}(p) = \inf\{t : p_t = p_e\} = \lim_{\widetilde{p}\to+\infty} T_{p_e,\widetilde{p}}(p)$$

Proposition 2. ***(Firms' Expected Life Span)*** *Denote $V(p) = T_{p_e,\widetilde{p}}(p)$. Using Feynman-Kac Theorem again, we list the fomulae for $T_{p_e,\widetilde{p}}$ and T_{p_e} as below.*

1.

$$\delta V(p) = 1 + V'(p)\mu_p p + \frac{1}{2}V''(p)\sigma_p^2 p^2$$
$$V(p_e) = V(\widetilde{p}) = 0$$

and thus

$$V(p) = \frac{1}{\delta} + v_1 p^{\gamma_1} + v_2 p^{\gamma_2}$$

where

$$v_1 \equiv (\frac{1}{\delta})(\frac{p_e^{\gamma_2} - \widetilde{p}^{\gamma_2}}{p_e^{\gamma_1}\widetilde{p}^{\gamma_2} - p_e^{\gamma_2}\widetilde{p}^{\gamma_1}})$$
$$v_2 \equiv (\frac{1}{\delta})\frac{\widetilde{p}^{\gamma_1} - p_e^{\gamma_1}}{p_e^{\gamma_1}\widetilde{p}^{\gamma_2} - p_e^{\gamma_2}\widetilde{p}^{\gamma_1}}$$

and $\gamma_1 < 0$ *and* $\gamma_2 > 1$ *denote respectively the small and large roots of the following quadratic equation.*

$$\frac{\sigma_p^2}{2}\gamma(\gamma - 1) = \delta - \gamma\mu_p$$

2. *Based on 1, immediately we have*

$$\lim_{\widetilde{p}\to+\infty} v_1 = -\frac{1}{\delta p_e^{\gamma_1}}$$
$$\lim_{\widetilde{p}\to+\infty} v_2 = 0$$

and thus

$$T_{p_e}(p) = \lim_{\widetilde{p}\to+\infty} V(p) = \frac{1}{\delta}[1 - (\frac{p}{p_e})^{\gamma_1}]$$

Corollary 3. *The expected life span of a firm $T_{p_e}(p)$ with $p \geq p_e$ has the following properties:*

1. *$T_{p_e}(p)$ is positively correlated with her current productivity p, $i.e.$, $\frac{dT_{p_e}(p)}{dp} > 0$, which is in line with the empirical findings in the literature on firm dynamics.*
2. *$T_{p_e}(p) < \frac{1}{\delta}$, where $\frac{1}{\delta}$ denotes the frims' expected life span with $\delta > 0, \mu_p = \sigma_p^2 = 0$.*
3. *$\delta T_{p_e}(p)$ is itself is an uncertain number, but it takes the form of Pareto distribution with parameter $-\gamma_1$.*

Substituting Proposition 1 into the free entry conditon $c = \theta q(\theta) \int_{\overline{p}-\sigma_{\overline{p}}}^{\overline{p}+\sigma_{\overline{p}}} (1-\beta) S(p) dG(p)$ yields that

$$c = \frac{\theta q(\theta)(1-\beta)}{2\sigma_{\overline{p}}} \int_{\overline{p}-\sigma_{\overline{p}}}^{\overline{p}+\sigma_{\overline{p}}} [\Xi(p) - \Xi(p_e)(\frac{p}{p_e})^{\alpha}] dp \tag{1}$$
$$= \theta q(\theta)(1-\beta)[a\overline{p} - b + D a^{\alpha} b^{1-\alpha}]$$

where

$$D \equiv (\frac{\alpha-1}{\alpha})^{\alpha} (\frac{1}{\alpha^2-1}) [\frac{(\overline{p}+\sigma_{\overline{p}})^{\alpha+1} - (\overline{p}-\sigma_{\overline{p}})^{\alpha+1}}{2\sigma_{\overline{p}}}] < 0, \text{ for any } \alpha \in (-\infty, -1) \cup (-1, 0)$$

Equation (1) can be rearranged and then rewritten as below.

$$LHS(\theta) \equiv \frac{c}{(1-\beta)\theta q(\theta)} + b$$
$$RHS(\theta) \equiv a\overline{p} + D a^{\alpha} b^{1-\alpha}$$
$$LHS(\theta) = RHS(\theta)$$

Using LHS and RHS yields the equilibrium level of θ, both of which are depicted in Figure 3. The existence and uniqueness of the solution is guaranteed if and only if

$$LHS(\widetilde{\theta}) < RHS(\widetilde{\theta})$$

where

$$-s + \frac{\beta c \widetilde{\theta}}{1-\beta} = 0$$

Equivalently, we need to assume that

$$\frac{c}{(1-\beta)\widetilde{\theta} q(\widetilde{\theta})} < a\overline{p}$$

That is, the entry cost c and the worker's bargaining power β should be small enough, and the average initial productivity $\overline{p}$ should be high enough, *ceteris paribus*.

Denote $\widehat{\theta}$ as the solution to $LHS(\theta) = RHS(\theta)$ and thus we get the steady-state variables for p_e, $w(p)$, *etc*.

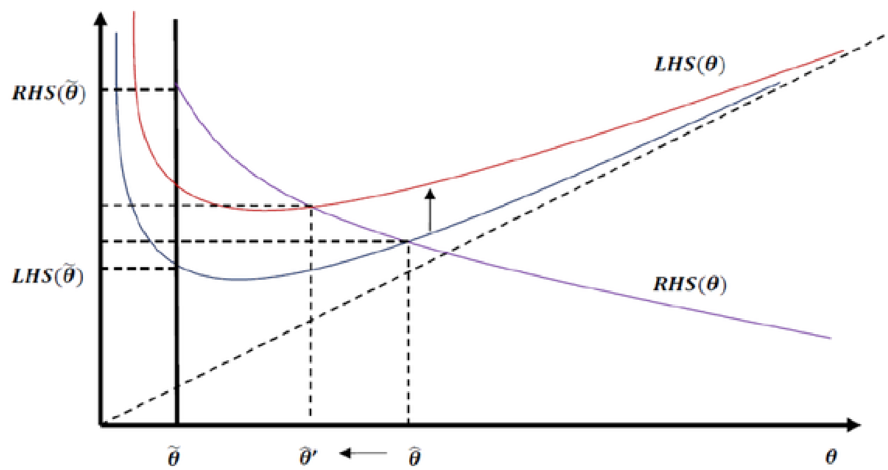


Figure 3: The Existence, Uniqueness and Comparative Statics (of c and β) of the Tightness of Labor Market $\widehat{\theta}$ in Equilibrium

Thus we need to make the following restriction to guarantee that $\overline{p} - \sigma_{\overline{p}} > p_e$ is always satisfied.

$$\overline{p} - \sigma_{\overline{p}} > (\frac{\alpha}{\alpha - 1})(\frac{r + \delta - \mu_p}{r + \delta})(-s + \frac{\beta c \widehat{\theta}}{1 - \beta})$$

Proposition 3. ***(Comparative Statics)****:*

1. *market tightness θ is negatively correlated with the posting cost c.*
2. *market tightness θ is also negatively correlated with worker's bargaining power β.*

The incentive for firm to post job declines as firm expects less profit from hiring: no matter it is because of high posting cost or a smaller share in the output claim.

3.1 Stationary Distribution

To better illustrate the problem of stationary distribution of the firm size, we define that

$$x \equiv \log(p), \widehat{x} \equiv \log(2\sigma_{\overline{p}}), x_e \equiv \log(p_e), p_L \equiv \overline{p} - \sigma_{\overline{p}}, p_H \equiv \overline{p} + \sigma_{\overline{p}}, x_L \equiv \log(p_L), x_H \equiv \log(p_H)$$

Since $p \sim G(p) = U[p_L, p_H]$, easily we know that x conforms to exponential distribution with the support $[x_L, x_H]$ and the probability density function $g(x)$ as below.

$$g(x) \equiv \exp(x - \widehat{x})$$

In the spirit of Dixit and Pindyck (1994), we approximate the Brownian motion with Random Walk. First of all, we make the following notations.

$$\mu_x \equiv \mu_p - \frac{1}{2}\sigma_p^2,\ \sigma_x \equiv \sigma_p,\ p_l \equiv \frac{1}{2}(1 - \frac{\mu_x}{\sigma_x}\sqrt{dt}),\ p_r \equiv \frac{1}{2}(1 + \frac{\mu_x}{\sigma_x}\sqrt{dt}),\ dh \equiv \sigma_x\sqrt{dt} \tag{2}$$

Additionally, we denote N as the entry rate of the potential entrants and $\phi(x)$ as the density of incumbents.

Now, let us characterize the distribution of incumbent firms $\phi(x)$. We will discuss the distribution on three intervals, $[x_e, x_L], (x_L, x_H], (x_H, +\infty)$ separately.

Case I When $x \in (x_L, x_H]$, the law of motion for measure of firm with productivity p is

$$Nq(\widehat{\theta})\phi(x)dh = Nq(\widehat{\theta})dtg(x)dh + p_r(1 - \delta dt)Nq(\widehat{\theta})\phi(x - dh)dh + p_l(1 - \delta dt)Nq(\widehat{\theta})\phi(x + dh)dh$$

Removing $Nq(\widehat{\theta})$ from both sides and using Taylor expansion yields

$$\frac{1}{2}\sigma_x^2\phi''(x) - \mu_x\phi(x) - \delta\phi(x) + g(x) = 0$$

It is easy to verify that the $\phi_0(x)$ is a particular solution to the above ODE, where $\phi_0(x)$ is defined as below.

$$\phi_0(x) = \frac{\exp(x - \widehat{x})}{\delta + \mu_x - \frac{1}{2}\sigma_x^2}$$

Assumption 2.1 insures we can always have $\phi_0(x) > 0$ since

$$\delta + \mu_x - \frac{1}{2}\sigma_x^2 = \delta + \mu_p - \sigma_p^2 > 0$$

Then the general solution to the above ODE is given by

$$\phi(x) = A_1 \exp(\xi_1 x) + A_2 \exp(\xi_2 x) + \phi_0(x) \text{ for } x \in (x_L, x_H]$$

where $\xi_1 < 0$ and $\xi_2 > 0$ are the roots of the following quadratic euqation,

$$\frac{1}{2}\sigma_x^2\xi^2 - \mu_x\xi - \delta = 0$$

and the constants A_1 and A_2 are to be determined.

Case II When $x \in [x_e, x_L]$, similarly, we can get the following ODE for $\phi(x)$ when $x \in (x_e, x_L]$.

$$\frac{1}{2}\sigma_x^2\phi''(x) - \mu_x\phi(x) - \delta\phi(x) = 0$$

Thus the solution can be written as below.

$$\phi(x) = B_1 \exp(\xi_1 x) + B_2 \exp(\xi_2 x) \text{ for } x \in [x_e, x_L]$$

where the constants B_1 and B_2 are to be determined.

Case III When $x \in (x_H, +\infty)$, the solving process is almost the same as the aforementioned second case, we can get the following ODE for $\phi(x)$ when $x \in (x_e, x_L]$.

$$\frac{1}{2}\sigma_x^2\phi''(x) - \mu_x\phi(x) - \delta\phi(x) = 0$$

Thus the solution can be written as below.

$$\phi(x) = C_1 \exp(\xi_1 x) + C_2 \exp(\xi_2 x) \text{ for } x \in (x_H, +\infty)$$

where the constants C_1 and C_2 are to be determined.

Boundary Conditions The unkowns A_1, A_2, B_1, B_2, C_1 and C_2 are determined by the following six boundary conditions.

$$\begin{aligned} \lim_{x\uparrow x_L}\phi(x) &= \lim_{x\downarrow x_L}\phi(x),\ \lim_{x\uparrow x_L}\phi'(x) = \lim_{x\downarrow x_L}\phi'(x) \\ \lim_{x\uparrow x_H}\phi(x) &= \lim_{x\downarrow x_H}\phi(x),\ \lim_{x\uparrow x_H}\phi'(x) = \lim_{x\downarrow x_H}\phi'(x) \\ \int_{x_H}^{+\infty}\phi(x)dx &< +\infty,\ \phi(x_e) = 0 \end{aligned}$$

As Karatzas and Shreve (1991) suggests, the first four restrictions can guarantee the sufficient smoothness of $\phi(\cdot)$. The fifth condition is imposed to ensure that the total mass of incumbents be finite. The last equation stems from the fact that x_e, to some extent, can be treated an *absorbing barrier*.[2]

$$
\begin{aligned}
B_1 \exp(\xi_1 x_e) + B_2 \exp(\xi_2 x_e) &= 0 \\
B_1 \exp(\xi_1 x_L) + B_2 \exp(\xi_2 x_L) &= A_1 \exp(\xi_1 x_L) + A_2 \exp(\xi_2 x_L) + \phi_0(x_L) \\
B_1\xi_1 \exp(\xi_1 x_L) + B_2 \exp \xi_2(\xi_2 x_L) &= A_1\xi_1 \exp(\xi_1 x_L) + A_2\xi_2 \exp(\xi_2 x_L) + \phi_0'(x_L) \\
A_1 \exp(\xi_1 x_H) + A_2 \exp(\xi_2 x_H) + \phi_0(x_H) &= C_1 \exp(\xi_1 x_H) \\
A_1\xi_1 \exp(\xi_1 x_H) + A_2\xi_2 \exp(\xi_2 x_H) + \phi_0'(x_H) &= C_1\xi_1 \exp(\xi_1 x_H) \\
C_2 &= 0
\end{aligned}
$$

Lemma 2. *The solutions to the above equation systems is listed in terms of p as below.*

$$
\begin{aligned}
A_1 &= \frac{(1-\xi_1)p_e^{\xi_2-\xi_1}(p_H^{1-\xi_2} - p_L^{1-\xi_2}) + (1-\xi_2)p_L^{1-\xi_1}}{(\xi_2-\xi_1)(p_H - p_L)(\delta + \mu_p - \sigma_p^2)} \\
A_2 &= \frac{(1-\xi_1)p_H^{1-\xi_2}}{(\xi_2-\xi_1)(p_H - p_L)(\delta + \mu_p - \sigma_p^2)} \\
B_1 &= \frac{(1-\xi_1)p_e^{\xi_2-\xi_1}(p_H^{1-\xi_2} - p_L^{1-\xi_2})}{(\xi_2-\xi_1)(p_H - p_L)(\delta + \mu_p - \sigma_p^2)} \\
B_2 &= \frac{(1-\xi_1)(p_L^{1-\xi_2} - p_H^{1-\xi_2})}{(\xi_2-\xi_1)(p_H - p_L)(\delta + \mu_p - \sigma_p^2)} \\
C_1 &= \frac{(1-\xi_1)p_e^{\xi_2-\xi_1}(p_H^{1-\xi_2} - p_L^{1-\xi_2}) + (\xi_2-1)(p_H^{1-\xi_1} - p_L^{1-\xi_1})}{(\xi_2-\xi_1)(p_H - p_L)(\delta + \mu_p - \sigma_p^2)} \\
C_2 &= 0
\end{aligned}
$$

Corollary 4. *The tightness of the labor market $\widehat{\theta}$ affects the value of optimal exiting rule p_e and thus has influence on A_1, B_1 and C_1.*

In the steady state, there exists a stationary distribution of surviving firms μ and a constant entry rate N. The following equation captures the evolution of the distribution of firm size/productivity.

$$\mu_{t+dt}(K) = (1-\delta dt)\int_{p^*}^{+\infty} Q(K|p)\mu_t(dp) + Nq(\widehat{\theta})G([p_L, p_H] \cap K)dt, \text{ for any Borel set } K$$

[2]At $x = x_e + dh$, we have the following equation,

$$\phi(x)dh = p_l(1-\delta dt)\phi(x+dh)dh.$$

Simplification shows that $\phi(x) = 0$. Letting dh go to zero, we have $\phi(x_e) = 0$.
In a nutshell, A_1, A_2, B_1, B_2, C_1 and C_2 are summarized in the following equation systems.

where $Q(\cdot|\cdot)$ denotes the transition function.

With the help of the last lemma, we have the following conclusion on the density function of the stationary distribution μ^*.

Proposition 4. ***(Stationary Distribution of Firm Size)*** *In terms of the productivity p, the density function of the stationary distribution μ^* is characterized as below.*

$$Nq(\widehat{\theta})\chi(p) = Nq(\widehat{\theta})\frac{\phi(\log(p))}{p}$$

where

$$\chi(p) = \begin{cases} B_1 p^{\xi_1 - 1} + B_2 p^{\xi_2 - 1}, & \text{if } p_e < p \leq p_L \\ A_1 p^{\xi_1 - 1} + A_2 p^{\xi_2 - 1} + \frac{1}{(p_H - p_L)(\delta + \mu_p - \sigma_p^2)}, & \text{if } p_L < p \leq p_H \\ C_1 p^{\xi_1 - 1}, & \text{if } p > p_H \end{cases}$$

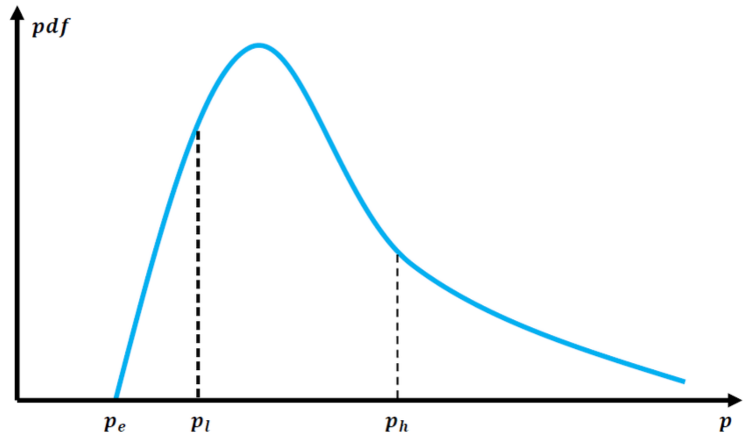
where A_1, A_2, B_1, B_2, C_1 are pinned down in the previous lemma.

Besides, the above proposition suggests that the stationary distribution of firm size/productivity is unimodal, right-skewed with a Paretian tail, which will be illustrated in the following corollary and Figure 4.

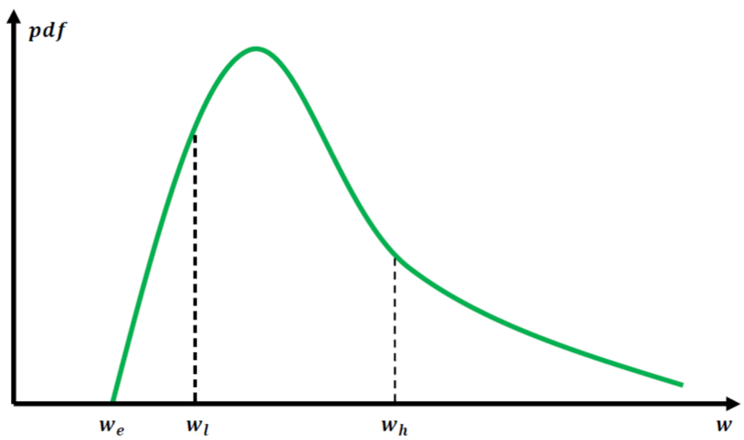
Corollary 5. ***(Zipf Law)*** *The firm-size distribution has a right-skewed Pareto-like tail, $i.e.$, for any $p \geq p_H$, the proportion of firm size being larger than p is $\eta p^{-\vartheta}$, where*

$$\vartheta \equiv -\xi_1 > 0$$
$$\eta \equiv -\frac{C_1}{\xi_1} > 0$$

Since $w(p) = \beta(p + c\widehat{\theta}) - (1 - \beta)s$, and by the above proposition, we can then easily get the probability distribution function of wage dispersion. We summarize the distribution of wage dispersion as well as that of firm size in the following figure.



(a) Distribution of Firm Productivity: Unimodal, Right-Skewed with a Paretian Tail



(b) Wage Dispersion: Unimodal, Right-Skewed with a Paretian Tail

Figure 4: Distribution of Firms

Moreover, we can detect the effect of the change of the entry cost c on the change of the distribution of firm size as well as on that of the distribution of wages.

3.2 Unemployment rate

Following the heuristic argument of using Random walk to approximate Brownian Motion in the previous subsection, we know that over dt the possibility ψ of which the productivity of incumbents hits x_e is given by

$$\psi = p_l(1 - \delta dt)Nq(\widehat{\theta})\phi(x_e + dh)dh$$

Substituting (2) and $\phi(x_e + dh) = \phi(x_e) + \phi'(x_e)dh$ into the above equation and using the facts that $x_e + dh \to x_e$ and $\phi(x_e) = 0$ yields that

$$\psi = \frac{1}{2} N q(\widehat{\theta}) \phi'(x_e) \sigma^2 dt$$

Thus the evolution of the umemployment rate can be written as below.

$$du = (1-u)\delta dt - u\widehat{\theta} q(\widehat{\theta}) dt + \frac{1}{2} N q(\widehat{\theta}) \phi'(x_e) \sigma_p^2 dt$$

In steady state, we have $du/dt \equiv \dot{u} = 0$. Thus the steady-state unemployment rate is given by

$$u = \frac{\delta + \frac{1}{2} N q(\widehat{\theta}) \phi'(x_e) \sigma_p^2}{\delta + \theta q(\widehat{\theta})}$$

In the stationary equilibrium, we have

$$(1-u)\delta dt = -\frac{1}{2} N q(\widehat{\theta}) \phi'(x_e) \sigma_p^2 dt + N q(\widehat{\theta}) dt$$

Removing dt from both sides yields that

$$(1-u)\delta = N q(\widehat{\theta})[1 - \frac{1}{2}\phi'(x_e)\sigma_p^2]$$

Therefore, given $(\delta, \sigma_p^2, \theta, \phi(\cdot), x_e)$, u and N are jointly determined by the following equation systems.

$$u = \frac{\delta + \frac{1}{2} N q(\widehat{\theta}) \phi'(x_e) \sigma_p^2}{\delta + \theta q(\widehat{\theta})}$$
$$(1-u)\delta = N q(\widehat{\theta})[1 - \frac{1}{2}\phi'(x_e)\sigma_p^2]$$

Thus we have

$$u = \frac{\delta + \zeta}{\delta + \zeta + \widehat{\theta} q(\widehat{\theta})}$$
$$N = \frac{(1-u)(1+\zeta)\delta}{q(\widehat{\theta})}$$

where

$$\zeta \equiv \frac{\frac{1}{2}\phi'(x_e)\sigma_p^2}{1-\frac{1}{2}\phi'(x_e)\sigma_p^2}$$

Immediately we get the aggregate output of the industry Q_A as below.

$$Q_A \equiv (1-u)\int_{p_e}^{+\infty} p\chi(p)dp$$

Proposition 5. ***(Transition Mechanism of Tightness θ on Unemployment Rate u)*** *The tightness of the labor market $\widehat{\theta}$ affects the unemployment through three channels. One is by the standard way, which is embeded in $\widehat{\theta}q(\widehat{\theta})$. The second channel is through $\widehat{\theta}$'s influence on the value of optimal exiting rule p_e (or equivalently, x_e) since we have shown that*

$$p_e = (\frac{\alpha}{\alpha-1})(\frac{r+\delta-\mu_p}{r+\delta})(-s+\frac{\beta c\widehat{\theta}}{1-\beta}) = \exp(x_e)$$

And finally, the third way is by the effect of $\widehat{\theta}$ on B_1 and B_2 and then affects the density function $\phi(\cdot)$ for $x \in [x_e, x_L]$.

3.3 Industry Dynamics

In the previous Section 2, we simply treat the market price P as exogenously given and then analyze the firm dynamics as well as the distribution of several variables of interest. In this Section, we are devoted to closing the model by considering the more general results in the level of industry dynamics and incorporating as below the demand side into the economy,

$$P = \Lambda Q^{-\frac{1}{\varepsilon}}$$

where Q the aggregate output of the industry, $\varepsilon > 0$ the price elasticity of demand and $\Lambda > 0$ is a constant demand coefficient defined as below.

$$\Lambda \equiv Q_A^{-\frac{1}{\varepsilon}}$$
$$Q_A \equiv (1-u)\int_{p_e}^{+\infty} p\chi(p)dp$$

Using the strategy of guess-and-verify, we have the following industry equilibrium.

Proposition 6. *In industry-level equilibrium, the market price $P = 1$ and other allocations and distri-*

butions are exactly the same as the those obtained in the firm-level equilibrium.

4 Conclusion

This paper proposes a tractable model to analyze the distribution of wages and firm size with one shot. The classic Mortensen and Pissarides (1994) is revistied in continuous time with the help of real option. Based on the model, we produce the exact distribution pattern of wage dispersion in the real-life data, which cannot be addressed by the influential Burdett-Mortensen's on-the-job-search model. Meanwhile, since we have double-sided search and matching story, our model also offers an analytical solution to the distribution of firm size as well as the detailed analysis on the firm's entry and exit decision. The distribution pattern predicted by our model is well in line with the documented Zipf law. Additionally, since we have integrated workers in the model, we lend more insight into the structural parameters on the distributions of wage and firm size by additionally considering the allocation of bargaining power, which has never been touched before. The comparative statics offer some possible sources for the secular change of the distributions of wage and firm size of the developed countries in the past decades.

However, the research project is far from perfect. First of all, for tractability in search and matching theory, we implicitly assume that each firm can *only* employ one worker. As a compromise, we employ the productivity (or the revenue) as the measurement of firm size. However, it seems more interesting and more realistic to use*the employment size* to measure the firm size. Maybe it is desirable to consider one-firm-multiple-worker extension in the future research. Secondly, although the model enjoys the quantitative fit of the distribution patterns of wage and firm size, it remains to be shown how the proposed model fit with the real data. Besides, it is potentially intriguing to fit our model into the data to *infer* the allocation and the evolution of bargaining power in the real world.

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