Discretion versus Rules without Common Knowledge*

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Abstract

This paper revisits the classical time inconsistent problem à la Barro and Gordon (1983) while relaxing the common knowledge assumption. When information is complete, achieving the first-best outcome is not possible, and welfare under committed rules surpasses that under discretionary policies. Conversely, in situations where information among agents is incomplete, a policymaker adhering to committed rules can always attain the first-best outcome. If a commitment device is unavailable, the resulting welfare loss can either increase, decrease, or potentially disappear. We quantify this insight in an economy with the inflation-unemployment tradeoff based on the survey evidence on expectations.

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1. Introduction

It is well known that time inconsistence is a pervasive issue in policy design. Crucially, the rational anticipations about future government policy in the private sector prevent the first-best outcome being implemented (Kydland and Prescott, 1977). In this paper, we revisit this classical problem. In contrast to the literature which focuses on repeated games and reputational forces (Barro and Gordon, 1983; Stokey, 1989; Chari and Kehoe, 1990), we restrict our attention to the static game but allow anticipations in the private sector to deviate from those formed under full information rational expectations (FIRE). Particularly, we explore whether the lack of common knowledge about others agents' beliefs alleviates or aggravates the time inconsistent problem.

FIRE benchmark. We consider a stylized framework similar to the model of inflation in Barro and Gordon (1983). The aggregate outcome depends on both a policy instrument and an endogenous action chosen by the private sector.[1] Subject to exogenous shocks to fundamentals, agents in the private sector choose their action according to their expectation about the policy instrument. The social cost results from both the discrepancy between the policy instrument and the fundamental and the discrepancy between the aggregate outcome and the fundamental.

With FIRE, even with the committed Ramsey policy rule, the first-best outcome—eliminating both discrepancies—may not be possible. In our setting, this is due to that the policy maker only have one policy tool and is unable to perfectly manipulate both the instrument and the aggregate outcome towards their targets. It is useful to note that whether the Ramsey policy rule depends on only the exogenous fundamental or also on the aggregate outcome is irrelevant. Other agents' action is common knowledge, and is proportional to the fundamental.

Meanwhile, when the policy maker choose the instrument in a discretionary fashion by acting upon agents' action, there is a further reduction of social welfare. The rational anticipation that the policy maker has incentive to deviate from the rule ex post alters the private sector's choice ex ante, leaving the time-consistent policy rule different from the Ramsey rule. This is the key insight in favoring commitment over discretion.

Lack of Common Knowledge. The rationality assumption not only requires that agents to be fully attentive to the changes in underlying fundamentals, but also to be fully aware of other agents' responses. The latter requirement is particularly strong and is at odds with the empirical regular-

[1] For example, the unemployment depends on households' inflation expectation and the actual path of inflation chosen by the central bank.

ities (Coibion and Gorodnichenko, 2015; Coibion, Gorodnichenko, Kumar, and Ryngaert, 2021). In this paper, we follow the shallow reasoning approach developed in Angeletos and Sastry (2021) which accommodates higher-order uncertainty: each agent perfectly observes the fundamental, but believes only a fraction of other agents are aware of the changes in fundamental. As a result, an individual agent perceives that others' expectations about the fundamental is less responsive compared to their own. Put it differently, the aggregate higher-order expectations are more dampened than first-order ones. This approach captures the key feature in models with dispersed noisy information (Lucas, 1972; Morris and Shin, 2002; Woodford, 2003) without the complication resulting from heterogeneous information sets.

The differential responses between first- and higher-order expectations imply that the functional form of the policy rule becomes relevant. If the policy rule is specified as a function of the fundamental alone, anticipation about the policy instrument involves only first-order expectations. In contrast, if the instrument is also contingent on the endogenous action, the anticipation about the instrument requires agents' inference about others' action, and therefore others' expectations about the fundamental, and so on. In this case, both first-order and higher-order expectations play a role in shaping the agents' actions.

From the perspective of the policy maker with commitment, the loading on the fundamental and on the endogenous action in the policy rule can influence agents' expectations in distinctive ways. Such additional flexibility enlarges the policy maker's choice set to the extent that the first-best allocation can be achieved. Note that the social welfare displays a form of discontinuity in the degree of bounded rationality: introducing a tiny amount of higher-order uncertainty can lead to discrete welfare improvement.

For a discretionary policymaker, such additional flexibility cannot be directly utilized as time-consistent policy rule imposes a specific . Since they act after the private sector, a unique ex-post policy rule for the instrument emerges. Such policy rule also relies on both the fundamental and the endogenous action, but may not necessarily move the implementability constraint closer to the targeted relative responsiveness in the social welfare function. However, we show that when desired responsiveness of the endogenous outcome is higher than that of the fundamental, the social welfare can be improved compared with the rational benchmark. Furthermore, even the first-best allocation can be achieved, which completely eliminates the cost of the lack of commitment.

2. Framework

Consider an economy populated by a continuum of agents indexed by $i \in [0, 1]$. Each agent chooses an action $k_i \in \mathbb{R}$. Let k denote the average action, that is, $k = \int k_i di$. The policy maker chooses her instrument τ, and we assume agents' best response is simply

$$k_i = E_i[\tau], \quad k = \overline{E}[\tau].$$

That is, agents try to align their action with the policy instrument. We use $\mathbb{E}_i$ to denote an agent's subjective expectation and $\overline{E}[\cdot] \equiv \int E_i[\cdot]di$ to denote average expectation. In the baseline specification, we abstract from the direct dependence on others' action, but our results are valid once we relax this assumption.

The aggregate outcome $y \in \mathbb{R}$ is determined jointly by the policy instrument τ and the aggregate action k

$$y = (1 - \alpha)\tau + \alpha k$$

We let α to measure the relative importance between the policy instrument τ and the aggregate action on the aggregate outcome y. In our analysis, we focus on the parameter space where $\alpha < 1$.

Policy game. Let $\theta \in \mathbb{R}$ be an exogenous stochastic fundamental. The social loss function is given

$$\mathcal{L} = \chi (y - \delta\theta)^2 + (1 - \chi)(\tau - \theta)^2.$$

The policy maker wishes to minimize the distance between the aggregate outcome and its target level $\delta\theta$, and the distance between the policy instrument and its target level that is normalized to θ. The parameter χ parameterizes the relative importance of meeting the two targets. As we will see momentarily, when $\delta \neq 1$, the desired responses of the aggregate outcome differs from that of the instrument, which is the root of the time inconsistency problem.

The key issue is that very often the private sector needs to make a decision before the policy instrument is actually determined. The firms may need to post nominal wages before the money supply is determined; investors may need to decide whether to expand their expenditure on structures and equipment before observing the capital income tax rate. How the private sector form such anticipation is the key in shaping the social welfare, and it is greatly influenced by the behavior of the policy maker.

In our setting, the game between the policy maker and agents unfolds as follows:

0. Before observing the fundamental, the policy maker announce the policy rule in setting the instrument.
1. The fundamental θ realizes. Each agent i forms expectations about the policy instrument and chooses their own action k_i.
2. The policy maker chooses the instrument τ, and the aggregate outcome and social welfare is determined.

We will consider two scenarios when setting the instrument in stage 2: the Ramsey policy with commitment and the discretionary policy that is time consistent. With the Ramsey policy, the instrument in stage 2 will be set according to the policy rule announced at stage 0 independent of the average action chosen in stage 1. With the discretionary policy, a policymaker can reset the instrument contingent on the average action.

Example. Before digging any further into the theory, we illustrate how our setting can nest some familiar economic problem. Consider the classical model of inflation à la Barro and Gordon (1983). Workers in the private sector forms their expectations about the inflation rate π, and the unemployment rate is determined via the Philips Curve

$$U = U^n - \beta(\pi - \overline{E}[\pi]),$$

where U is the unemployment rate and U^n is the natural rate of unemployment rate.

The social loss function can be expressed as

$$\mathcal{L} = U^2 + (\pi - \pi^*)^2,$$

where π^* is an exogenous inflation target.

In this environment, the policy instrument is the inflation rate π, the individual's action is simply $k_i = E_i[\pi]$, the aggregate outcome that depends on the instrument and average action is the unemployment rate. Here, we do not impose that everyone shares the same belief and therefore what enters the Phillips curve is the average expectation.

The policymaker aims to strikes a balance between the a higher unemployment rate and a deviation from the exogenous inflation target. The time inconsistent issue arises as the policy maker has an incentive to revise their plan to crease an inflation surprise ex pose in order to lower the unemployment rate.

3. Rational Expectations Benchmark

In this section, we present the equilibrium outcomes when agents are with full information and rational expectations (FIRE). These findings establish benchmarks for comparison with our main analysis without common knowledge in the next section.

The FIRE assumption ensures that agents have perfect inference about the fundamental as well as others' action

$$E_i[\theta] = \theta, \quad \text{and} \quad E_i[k] = k.$$

The former comes from the fact that agents can perfect observe the economic fundamental or the absence of first-order uncertainty, while the latter relies on the fact that all agents share the same information and their is no higher-order uncertainty.

Ramsey policy. We first consider the optimal policy under commitment. The policymaker has to commit to a particular policy rule ex ante and wish to minimize the expected social loss $\mathbb{E}[\mathcal{L}]$. The problem can be written as

$$\min_{\tau} \mathbb{E}\left[\chi\,(y-\delta\theta)^2 + (1-\chi)\,(\tau-\theta)^2\right],$$

subject to

$$k = \int E_i[\tau], \qquad y = (1-\alpha)\tau + \alpha k.$$

Under FIRE, it is immediate that $k = E_i[\tau] = \tau$ and the the policymaker takes this into account. The first-order condition satisfies

$$\chi\,(y-\delta\theta) + (1-\chi)\,(\tau-\theta) = 0. \tag{1}$$

The following proposition characterizes the outcome under the Ramsey policy.

Proposition 3.1. *With commitment, the equilibrium outcome has the following property.*

1. *The Ramsey policy takes the following form*

$$\tau = \mu\theta + \varphi k,$$

where (μ, φ) *satisfy*

$$\frac{\mu}{1-\varphi} = \chi\delta + (1-\chi). \tag{2}$$

Particularly, it is optimal for the policy to only react to the fundamental, $\varphi = 0$*.*

2. *When* $\delta \neq 1$*, the expected social loss is positive.*

Under FIRE, the aggregate outcome always equals the instrument, $y = \tau$, which the implementability constraint faced by the policymaker. Therefore, it is not possible to simultaneously set $y = \delta\theta$ and $\tau = \theta$ when $\delta \neq 1$. Optimally, the policymaker strikes a balance between the two targets, yielding $\tau = \chi\delta\theta + (1-\chi)\theta$. This logic is illustrated in Figure 1, where the optimal policy selects the point that satisfying the implementability constraint and is closes to the target $(\delta\theta, \theta)$.

Meanwhile, whether the policy rule is specified in terms of the fundamental or the aggregate action is irrelevant given that the ideal combined responses. As the average action is proportional to the fundamental and agents share the same belief, forming expectations about the average action is the same as that about the fundamental. The form of the policy rule plays no role in how agents anticipate the instrument.

Discretionary policy. Now we consider an alternative policy environment where the policymaker cannot honor the promised policy rule and can act upon the private sector's action ex post. Their problem becomes

$$\min_{\tau} \chi (y - \delta\theta)^2 + (1-\chi)(\tau - \theta)^2,$$

subject to

$$y = (1-\alpha)\tau + \alpha k.$$

Notice that both the fundamental and the average action have been realized and the policymaker now takes both of them as given. The first-order condition changes to

$$\chi (y - \delta\theta)(1-\alpha) + (1-\chi)(\tau - \theta) = 0. \tag{3}$$

Compared with condition (1), an additional term $(1-\alpha)$ shows up. This captures the fact that the average action k is not responding to the newly set instrument. The optimal choice of τ now is characterized by the following time-consistent policy rule.

Proposition 3.2. *The discretionary policy rule takes the form of $\tau = \mu^D \theta + \varphi^D k$ where*

$$\mu^D = \frac{\chi(1-\alpha)\delta + (1-\chi)}{\chi(1-\alpha)^2 + (1-\chi)}, \quad \varphi^D = -\frac{\chi\alpha(1-\alpha)}{\chi(1-\alpha)^2 + (1-\chi)},$$

where $\varphi < 0$ when $\alpha \in (0,1)$ and $\varphi > 0$ when $\alpha < 0$.

Different from the Ramsey policy where a continuum of policy rule indexed by (μ, φ) deliver the same outcome, the time-consistent policy takes a unique form. Meanwhile, note that this policy rule is independent of how agents form expectations, as the private sector moves first and cannot respond to the choice of τ afterwards.

However, with rational expectations, agents in the private sector perfectly anticipate that the only credible policy rule the policymaker will employ is the time-consistent one. Notice that the pair (μ^D, φ^D) does not satisfy condition (2), which imply that in responding to the time-consistent policy rule necessarily yields higher social loss. This is the paradox: even if their intention is to reduce social loss, the anticipation of this inconsistency by the agents can lead to an equilibrium outcome less favorable than steadfast commitment.

Proposition 3.3 (Commitment Efficacy)**.** *The discretionary policy yields allocation satisfying*

$$\tau = y = \frac{\mu^D}{1-\varphi^D}\theta.$$

A policy implemented with the Ramsey policy invariably results in a smaller social loss compared to the discretionary one.

This is visually captured in Figure 1, where Panel 1a and 1b depicts the scenarios with $\delta > 1$ and $\delta < 1$, respectively. The blue line represents implementability constraint $\tau = y$, which applies to both the Ramsey policy and the discretionary policy. With the Ramsey policy, satisfying $y = \tau$ is the only constraint faced by the policymaker, and the blue points corresponding to the ones that minimizes the distance with the targets. In contrast, with discretionary policy, the incentive to minimize the social loss ex post regulates that the the allocation has to satisfy the additional condition (3) as well, which is represented by the red dotted line. As a result, the discretionary policy yields the allocation that is further away from the target.

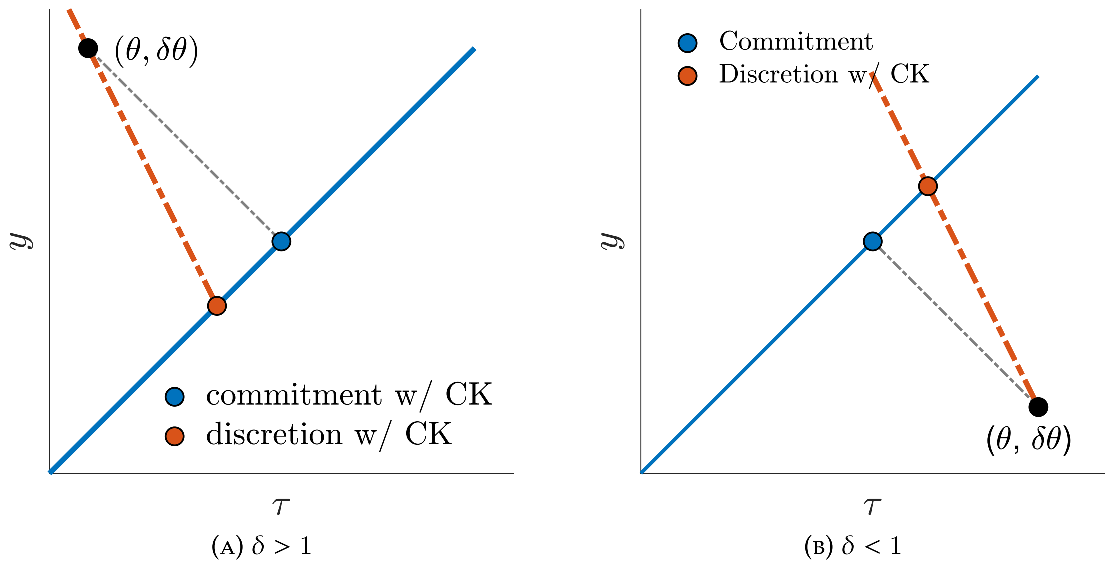


FIGURE 1: Commitment Efficacy under REE, assume $\chi = 0.5$

4. Optimal Policy with Incomplete Information

In this section, we present our main results on the optimal policy without imposing common knowledge about others' responses. We explore the welfare consequence when the strong notation of rationality is relaxed.

Higher-order expectations. It is useful to revisit how the expectations about the policy instrument is formed. For any individual agent, they take the policy rule $\tau = \mu\theta + \varphi k$ as given, and their best response can be expressed as

$$k_i = \mu E_i[\theta] + \varphi E_i[k].$$

That is, the allocation satisfies a beauty contest game where the parameters in this game are chosen by the policymaker. Iterating this beauty contest yields

$$k_i = \mu E_i[\theta] + \mu \sum_{n=1}^{\infty} \varphi^n E_i[\overline{E}^n[\theta]], \tag{4}$$

where $\overline{E}^n[\cdot]$ denotes the n-th order average expectation. Aggregating condition (4) implies that

$$k = \mu \overline{E}[\theta] + \mu \sum_{n=1}^{\infty} \varphi^n \overline{E}^{n+1}[\theta]. \tag{5}$$

This expression makes it clear that the average action depends on both the first-order expectation and higher-order expectations about the fundamental.

The FIRE assumption imposes two requirements: (1) agents are both rational and aware regarding their own rationality and attentiveness; (2) there is an universal understanding that all aspects are common knowledge, concerning beliefs about others. The first requirement leads to $E_i[\theta] = \overline{E}[\theta] = \theta$, and the second requirement leads to $\overline{E}^n[\theta] = \overline{E}[\theta]$ for all n. That is, all the higher-order expectations coincide with the first-order ones and are perfectly responsive to the fundamental. As a result, from agents' perspective,

$$k = \frac{\mu}{1-\varphi}\theta,$$

the distinguish between first-order and higher-order expectations are irrelevant and only the combined effects $\frac{\mu}{1-\varphi}$ in the policy rule matters.

Higher-order doubts. Our main analysis deviates such strong notion of rationality. Particularly, we following Angeletos and Sastry (2021) and discard the second requirement.

Assumption 4.1 (Higher-order doubts)**.** 1. *Each agent observes the fundamental perfectly.*

2. *Each agent believes that only a fraction of λ agents who observe the fundamental perfectly, and the rest $1-\lambda$ agents are not aware of the changes in fundamental.*

Assumption 4.1 introduces a wedge between first-order expectations and higher-order expectations. Note that when forming expectations about others,

$$\overline{E}[\theta] = \theta, \qquad E_i[\overline{E}[\theta]] = \lambda\theta.$$

The expectation about others' expectation is dampened due the concern that others may not be fully aware of the changes in fundamental. Such pattern that higher-order expectations are more dampened than first-order ones is consistent with the key properties in models with rational agents and dispersed information (Lucas, 1972; Morris and Shin, 2002; Woodford, 2003), and is also present with level-K thinking (Farhi and Werning, 2019; Iovino and Sergeyev, 2023; Bianchi-Vimercati, Eichenbaum, and Guerreiro, 2023). It is also consistent with the empirical evidence on expectations (Coibion and Gorodnichenko, 2015; Coibion, Gorodnichenko, Kumar, and Ryngaert, 2021). The higher-order doubt approach captures these features in a tractable way.

Note that the parameter λ quantifies the magnitude of deviation from rationality. At $\lambda = 1$, it reverts to the FIRE benchmark. If $\lambda = 0$, agents behave as if that nobody knows fundamental except for themselves. A reduction in λ intensifies the lack of common knowledge.

What are the implications of the deviation from common knowledge on the expectations about the policy instrument? The average higher-order expectations now obey

$$\overline{E}^n[\theta] = \lambda^{n-1}\theta, \quad \text{for} \quad n > 1.$$

As the order of expectations increases, the response to the fundamental is further dampened. The beauty contest (5) now becomes

$$k = \mu\theta + \mu \sum_{n=1}^{\infty} \varphi^n \lambda^n \theta = \frac{\mu}{1 - \varphi\lambda}\theta. \tag{6}$$

Notice that for a given policy rule, the response to the average action φ now plays a different role than that to the fundamental μ. It is the interaction product between φ and λ that matters. This is because the higher-order doubt leaves the partial equilibrium consideration or the expectations about θ unchanged, but only arrests the general equilibrium consideration about others' behavior. Next, we explore its impact on the policy design.

4.1 Shift of Implementability Constraint

Recall that with FIRE, the implementable pairs were given by $\{(\tau, y) : y = \tau\}$ independent of the policy rule. With higher-order doubts, equation (6) reveals that the average action k and the policy instrument τ will be different from each other. Therefore, the implementability constraint will be shifted as well.

Lemma 4.1. *Assume agents take a policy rule indexed by (μ, φ) as given. The set of pairs (τ, y) that are implementable is*

$$\mathcal{A}(\varphi) = \left\{(\tau, y) : \quad y = \left(1 - \alpha + \frac{\alpha}{1 + \varphi(1 - \lambda)}\right)\tau\right\}.$$

Ramsey policy. Lemma 4.1 shows that when $\lambda \in [0, 1)$, the implementability constraint becomes $y = \kappa\tau$ where the slope κ differs from 1 and varies φ. Note that in the first best, $y = \delta\tau$ but the policymaker cannot make this happen under FIRE and only a second-best allocation is obtainable. In contrast, with higher-order doubt, a careful choice of φ can lead to $y = \delta\tau$. In addition, the policy parameter μ can be chosen separately so that the sensitivity of τ equals to θ. The following result is then immediate.

Proposition 4.1. *With commitment, the policymaker can implement the first-best allocation $y = \delta\theta$ and $\tau = \theta$ by setting*

$$\mu = 1 + \frac{\delta - 1}{\alpha(1-\lambda)}, \quad \textit{and} \quad \varphi = \frac{\delta - 1}{(1-\alpha-\delta)(1-\lambda)}.$$

In contrast with the equilibrium outcome under FIRE, the first-best allocation is implementable. With FIRE, μ and φ do not affect the outcome separately and only the combination $\frac{\mu}{1-\varphi}$ matters. That is, there is a one degree of freedom to influence the outcome. With higher-order doubt, by shifting φ, the policymaker has the ability to manipulate the relative weight between first-order and higher-order expectations. This gives the policymaker an additional degree of freedom which allows for the first-best outcome. As a by product, it also implies that by only responding to the fundamental alone, $\varphi = 0$, the policy maker can only implement the original allocation under FIRE.

Another remark is that even with just a tiny bit of higher-order uncertainty, the Ramsey policy can implement the first-best allocation. Therefore, the social welfare displays a form of discontinuity in the degree of rationality. That being said, it is useful to note that when λ is close to 1, the sensitivity of the policy instrument towards k and θ approaches infinity.

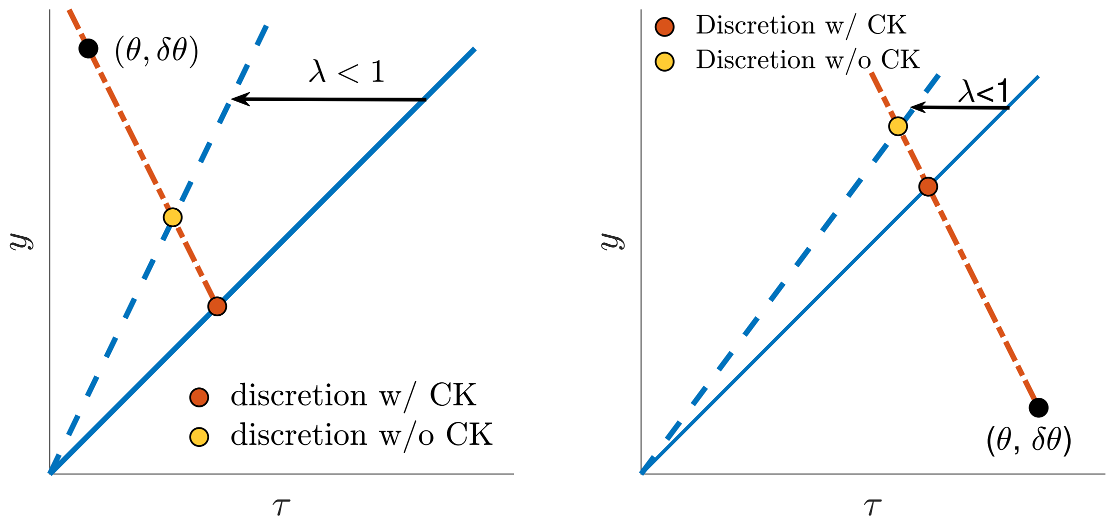
Discretionary policy. How does the lack of common knowledge alter the outcome for discretionary policy? Although the presence of higher-order doubt alters the slope of the implementability constraint, it requires specific policy rule so that the altered slope can yield higher social welfare. Note that with discretion, the policymaker only adopts the time-consistent rule characterized by (μ^D, φ^D). It turns out that the slope of the implementability constraint always becomes steeper.

Proposition 4.2. *With time-consistent policy rule, the implementable outcomes are given by*

$$\mathcal{A}(\varphi^D) = \{(\tau, y): \quad y = \xi(\lambda)\tau\}, \quad \textit{where} \quad \xi(\lambda) = 1 - \frac{\alpha\varphi^D(1-\lambda)}{1+\varphi^D(1-\lambda)}.$$

The slope $\xi(\lambda) > 1$ for any $\lambda \in [0, 1)$ and $\xi'(\lambda) < 0$.

In the context of a discretionary policy, the aggregate outcome y responds more than one-to-one



(A) $\delta > 1$

(B) $\delta < 1$

FIGURE 2: implementability constraint for discretionary policy

to τ, and the implementability is steeper as higher-order uncertainty intensifies. Interestingly, such result is independent of the sign of α.

Let's first examine the scenario where $\alpha > 0$. Recall from Proposition 3.2, the policy instrument exhibits a negative response to the collective action ($\varphi^D < 0$). The individual's optimal response, represented by $k_i = \mu^D E_i[\theta] + \varphi^D E_i[k]$, suggests that agents are engaged in a beauty contest game characterized by strategic substitution. Consequently, agents strive to act counter to the majority. When higher-order doubts come into play, each agent assumes that others will exhibit a muted reaction. Consequently, to distinguish themselves, agents are more reactive to their unique information. As this line of reasoning is universally adopted, it culminates in an exaggerated response in k, thereby magnifying y's sensitivity to fundamental shocks.

Now, consider the alternative, where $\alpha < 0$. In this scenario, the policy displays a positive response to k, implying $\varphi^D > 0$. Consequently, the best response of each agent is to move in a strategically complementary way. With the dampening effect derived from the higher-order doubts, individual agent choose to under-react to their signals. Although this leads to under-reaction in k, the fact that k is undermining aggregate outcome ($\alpha < 0$) makes y responds more than one-to-one to the fundamental shock.

Figure 2 visualizes the influence of higher-order doubts on the equilibrium outcome. Specifically, Panel 2a presents the scenario for $\delta > 1$, while Panel 2b depicts the scenario for $\delta < 1$. The blue solid line still represents the implementability constraint under common knowledge, while the blue dashed line represents the implementability constraint with higher-order doubt. Notably, as the intensity of higher-order doubts amplifies (i.e., λ diminishes), the blue dashed line pivots around the origin and becomes increasingly steep. The red dotted line represents the incentive constraint (3) as before, and its intercept with the implmentability constraint identifies the equilibrium outcome.

Whether the rotation of the implementability constraint favors a better outcome depends on whether δ is larger than or less than 1. If δ is larger than 1, additional response of y is preferred and higher-order doubt helps induce such reaction. In contrast, when $\delta < 1$, the rotation of the implementability constraint necessarily imply a lower social welfare.

4.2 Welfare Analysis

In this section, we formalize the welfare comparison hinted in the previous section.

Proposition 4.3. *With discretionary policy and higher-order uncertainty*

(i) welfare improves over common knowledge case when $\delta > 1$;

(ii) welfare worsens than common knowledge case when $\delta < 1$;

This result calls for special attention when studying the welfare cost of the lack of commitment. Without common knowledge, the welfare loss may be reduced or amplified depending on the socially desired relative responsiveness δ.

A useful benchmark for comparison is whether the welfare under discretionary policy without common knowledge can be higher than the Ramsey outcome with common knowledge. Also recall the Ramsey outcome with common knowledge is the same as that without common knowledge when restricting the policy rule to only respond to the fundamental.

Proposition 4.4. *The discretionary policy can achieve a higher welfare than the restricted Ramsey outcome without responding to average action if and only if the following condition is satisfied*

$$0 < \delta - 1 < -\frac{\alpha\varphi^D}{1+\varphi^D-\Delta}, \quad \textit{where } \Delta = \sqrt{\chi\left(1-\alpha\right)^2 + 1 - \chi}.$$

Furthermore, there exists $\lambda \in [0, 1)$ so that the discretionary policy can achieve the first-best allocation when

$$0 < \delta - 1 < -\frac{\alpha\varphi^D}{1+\varphi^D}.$$

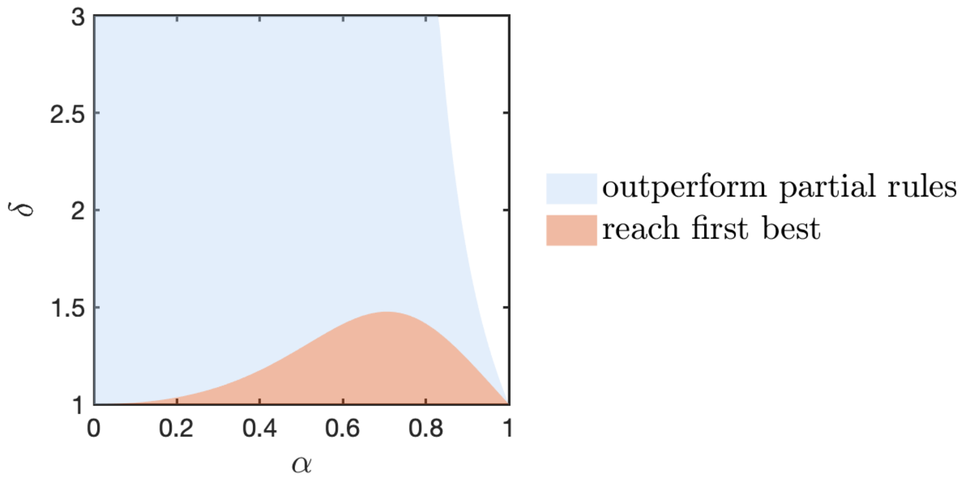


FIGURE 3: Lack of Commitment improves welfare over partial commited rules

Notably, when $\delta > 1$, not only the discretionary policy rule can outperform the Ramsey outcome with common knowledge, but also the improvement of welfare may even reach the first-best outcome for some level of higher-order doubt.

Figure 3 illustrates Proposition 4.4 by plotting the feasible sets of parameter pair in (α, δ) space that lead the discretionary policy to either outperform the fundamental commitment policy or achieve first best. The red shaded area corresponds to the parameter combination where first best can be obtained. The red shaded area plus the blue shaded area corresponds to the parameter combination where the discretionary policy can improve over fundamental commitment rules.

Finally, Corollary 4.1 characterizes how the social loss under discretionary policy varies with the degree of higher-order doubt λ.

Corollary 4.1.

1. *When* $0 < \delta - 1 < -\alpha \frac{\varphi^D}{1+\varphi^D}$, *the social loss first decreases then increases in* λ.
2. *When* $\delta - 1 > -\alpha \frac{\varphi^D}{1+\varphi^D}$, *the social loss is monotonically increasing in* λ.
3. *When* $\delta - 1 < 0$, *the social loss is monotonically decreasing in* λ.

5. Applications

In this section, we consider a variation of the inflation examples and quantify the effects of higher-order doubts. We show that in this example, the presence of higher-order uncertainty provides rationale for policymakers to favor a discretionary approach in monetary policy decisions.

5.1 Optimal Monetary Policy with Mark-up Shock

The objective of the policymaker is to minimize the following social loss:

$$\min_{\pi_t} \ \pi_t^2 + \chi(x_t - \theta_t)^2,$$

where π_t is inflation, x_t is the output gap, and θ_t corresponds to the mark-up shock shock. The policy maker faces the following static Philips Curve

$$\pi_t = \beta\overline{\mathbb{E}}_t[\pi_t] + \kappa x_t.$$

Relating this to our overarching model, the policy instrument τ_t corresponds to π_t, the outcome y_t is synonymous with x_t, and the average action k_t aligns with $\overline{\mathbb{E}}_t[\pi_t]$. The static Philips Curve implies that the aggregate outcome x is determined jointly by the policy instrument π and the aggregate action $\mathbb{E}[\pi]$,

$$x_t = \frac{\pi - \beta\overline{\mathbb{E}}[\pi]}{\kappa}$$

In this framework, δ is effectively infinite, indicating the policymaker's preference for only the output gap, and not inflation, to fluctuate with the mark-up shock θ_t. To assess the model in a dynamic context, we expand upon our prior framework to introduce a persistent fundamental process and facilitate learning among agents.

We posit that θ_t adheres to an AR(1) process, with ρ denoting the persistence parameter and ϵ representing the innovation:

$$\theta_t = \rho\theta_{t-1} + \epsilon_t$$

Given that agents can learn from historical data, their higher-order uncertainties about past events wane over time. Specifically, every agent i has an accurate perception of both past and present shocks ϵ_{t-k}. However, agent i assumes that only a fraction λ_k of their counterparts recognize ϵ_{t-k}, the disturbance from k periods earlier. Intriguingly, as k increases, implying a more distant shock, agent i presumes a growing proportion of individuals are informed about it: λ_k increases in k and approaches 1 as k becomes large. To encapsulate this notion, we define the sequence $\{\lambda_k\}$ as:

$$\lambda_k = 1 - \omega^k$$

The parameter ω dictates the initial magnitude of higher-order doubt (represented by $\lambda_1 = 1-\omega$) as well as the pace at which agents learn. Figure 4 illustrates two sample trajectories for the λ_k sequence. When learning occurs rapidly (indicated by a smaller ω), the initial value of λ is elevated and it converges to 1 more swiftly. Conversely, with slower learning, agents require a lengthier time to recognize that their peers are also aware of the shock.

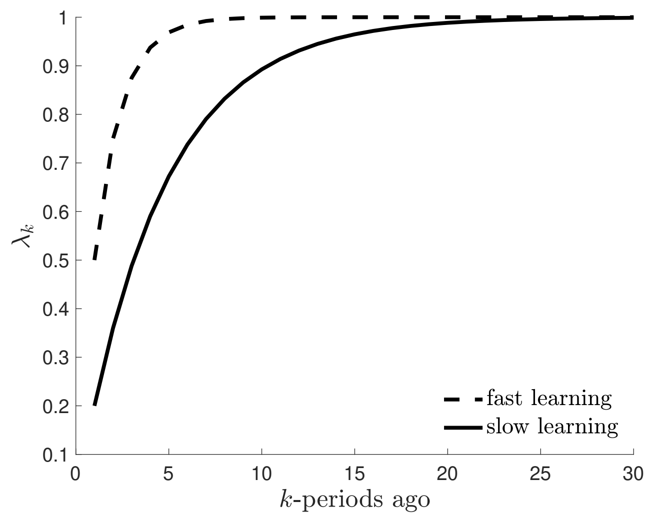


FIGURE 4: Learning trajectory with different ω

Calibration. We set some standard parameters as follows: χ, the weight on output deviation, is set at 0.5 (though the central conclusions remain consistent even if this parameter varies). β, representing the discount rate, is fixed at 0.99. Furthermore, κ, denoting the slope of the Philips curve, is established at 0.3.

The most important parameter in this exercise is ω, which indicates the level of higher-order uncertainty. We calibrate this parameter based on the evidence presented by Coibion, Gorodnichenko, Kumar, and Ryngaert (2021). In their survey, firm managers provide their forecasts about inflation, corresponding to first-order expectations. Importantly, managers also provide their belief regarding the average expectations of inflation, aligning with higher-order expectations in our framework. They find that a regression of higher-order expectation on the first-order expectation yielded a coefficient of 0.66. This suggests that higher-order expectations are more dampened:

$$\mathbb{E}_{it}[\overline{\mathbb{E}}_t[\pi_t]] = \underbrace{K}_{0.66} \mathbb{E}_{it}[\pi_t] + \text{residual}.$$

Incorporating their empirical observations, we select a value for ω such that, within our model, the dampening of higher-order expectation mirrors the extent observed in the data.

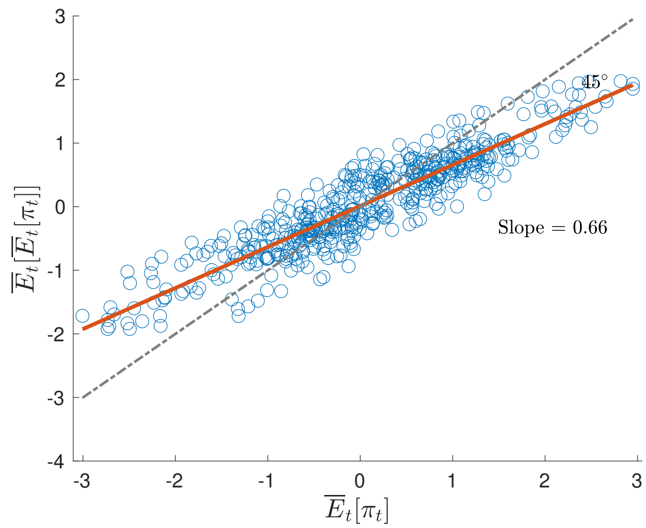


FIGURE 5: First-Order versus Second-Order Expectations

Figure 5 displays our calibration results. The horizontal axis represents the inflation expectation, corresponding to the first-order expectation, while the vertical axis denotes the agent's expectation of an average agent's expectation, synonymous with the higher-order expectation. By executing a regression in our model economy, the obtained sample of both first-order and higher-order expectations mirrored the same gradient.

In the case where the policy maker can commit, we assume that the Ramsey policy depends on current fundamental and current average inflation expectation,

$$\pi_t = m\theta_t + n\overline{\mathbb{E}}_t[\pi_t].$$

Iterating this policy generates the following representation,

$$\pi_t = m\sum_{k=0}^{\infty}\rho^k\theta_{t-k} + n\overline{\mathbb{E}}\left[\overline{\mathbb{E}}_t[\pi_t]\right].$$

The equilibrium average inflation expectation, inflation and output gap are respectively given by

$$\overline{\mathbb{E}}_t[\pi_t] = m\sum_{k=0}^{\infty}\frac{\rho^k}{1-n\lambda_k}\theta_{t-k},\ \pi_t = m\sum_{k=0}^{\infty}\left(1+\frac{n}{1-n\lambda_k}\right)\rho^k\theta_{t-k},\ x_t = \frac{m}{\kappa}\sum_{k=0}^{\infty}\left(1+\frac{n-\beta}{1-n\lambda_k}\right)\rho^k\theta_{t-k}.$$

The reduced form loss function under Ramsey Policy is given by

$$\mathcal{L} = \sum_{k=0}^{\infty}\left(m\left(1+\frac{n}{1-n\lambda_k}\right)\right)^2\rho^{2k} + \chi\left[\sum_{k=0}^{\infty}\left(\frac{m}{k}\left(1+\frac{n-\beta}{1-n\lambda_k}\right)-1\right)^2\rho^{2k}\right].$$

Now turn to the case where the policy maker adopts the discretionary policy,

$$\pi_t = m^D\theta_t + n^D\overline{\mathbb{E}}_t[\pi_t].$$

The optimality condition taking $\mathbb{E}_t[\pi_t]$ as given yields

$$m^D = \frac{\chi\kappa}{\chi+\kappa^2}, \quad n^D = \frac{\chi\beta}{\chi+\kappa^2}.$$

The reduced form loss function under discretionary Policy is given by

$$\mathcal{L} = \sum_{k=0}^{\infty}\left(m^D\left(1+\frac{n^D}{1-n^D\lambda_k}\right)\right)^2\rho^{2k} + \chi\left[\sum_{k=0}^{\infty}\left(\frac{m^D}{k}\left(1+\frac{n^D-\beta}{1-n^D\lambda_k}\right)-1\right)^2\rho^{2k}\right].$$

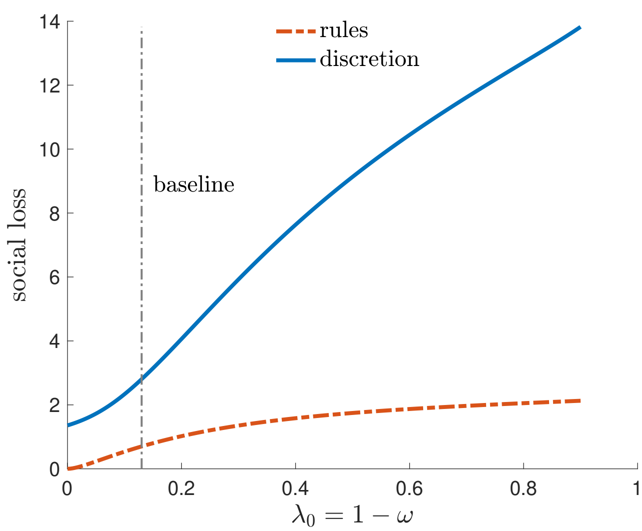
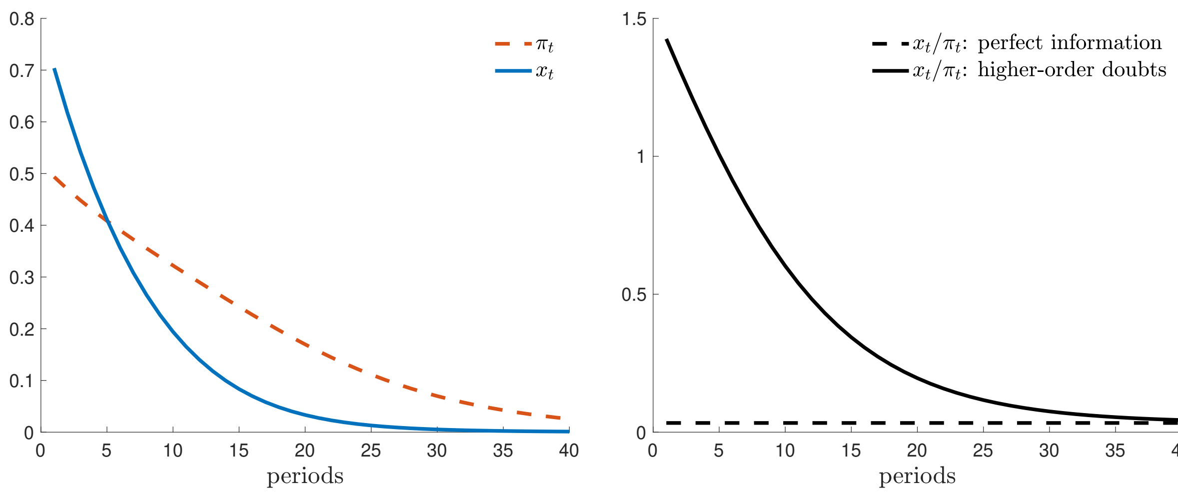


FIGURE 6: Welfare Comparison without Common Knowledge

In Figure 6, we observe the reaction of social welfare to heightened higher-order uncertainty. The horizontal axis, denoted by λ_0, signifies the perceived fraction of agents who recognize the current shock. As λ_0 approaches 1, it reverts to a common knowledge setting, while nearing 0 signifies a peak in the ”crisis of confidence”. The blue curve represents the social loss from adopting discretionary policy and the red curve represents that from committing to Ramsey policy. Two key insights emerge: First, as deviations from common knowledge increase, welfare improves under both the commitment and discretionary policies, consistent with our baseline theoretical results. Second, the welfare gain is more pronounced for the discretionary policy as higher-order uncertainty grows. This suggests that heightened higher-order uncertainty diminishes the policy maker's incentive to install a commitment device.

Role of Learning. In Figure 7, the impulse response function of inflation and the output gap under a discretionary policy is illustrated. Initially, the output gap, x_t, reacts more than inflation, π_t. However, over time, the responsiveness of x_t diminishes compared to π_t. The right panel showcases the x to π ratio. At the outset, this ratio is high, aligning with the policymaker's preference for it to approach infinity. Yet, as higher-order uncertainty resolves over time, the ratio reverts to its baseline level associated with common knowledge.



(A) Impulse responses

(B) Implied implementability

FIGURE 7: Role of learning

Slope of the Phillips Curve. Another important parameter in this exercise is the slope of Philips curve, especially when evaluating the welfare loss associated with a policymaker's discretionary policy. Recent empirical data underscore a gradual flattening of the Phillips curve over time. Understanding the implication of this trend within our model's framework is crucial.

Notice that the degree of complementarity φ^D in our policy game, is inversely related to the

Phillips curve's slope, κ.

$$\varphi^D = \frac{\alpha\beta}{\alpha + \kappa^2}$$

A flatter Phillips curve signifies increased strategic complementarity among agents.

In Figure 8, the blue line corresponds to the case with the steeper Philips curve and the red line corresponds to the case with the flatter Philips curve. We can draw two observations from this figure: First, in environments underpinned by common knowledge, a more flattened curve correlates with heightened welfare loss. Second, this flattening also amplifies the effects of higher-order uncertainty. As the higher-order uncertainty grows, the influence of κ on social welfare magnifies, leading to more pronounced welfare gains when moving away from a common knowledge foundation.

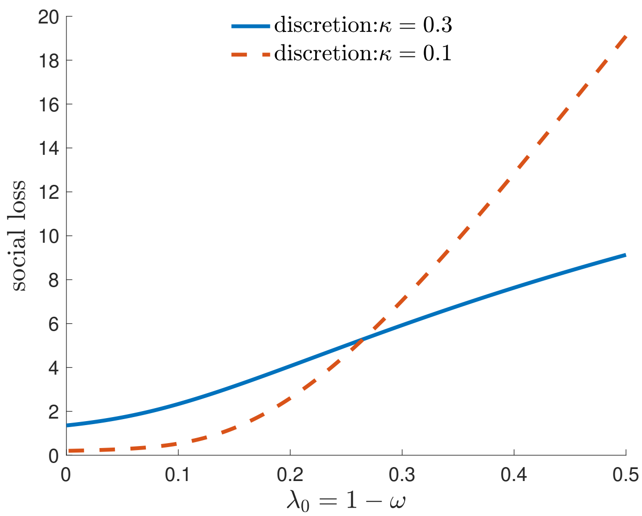


FIGURE 8: Welfare and Slope of Phillips Curve

6. CONCLUSION

In conclusion, the absence of common knowledge empowers policymakers by allowing them to shape both first and higher-order beliefs. As opposed to common knowledge environment, the presence of higher-order uncertainty will invariably leads to enhanced welfare if the policymaker make holistic commitment. The same cannot be said for a discretionary policy, which may, under certain conditions, result in its deterioration.

This underscores the importance of precisely identifying the root of time inconsistency. Such identification is crucial when policymakers are faced with the choice between adopting a discretionary policy or adhering to a committed policy rule. Making the right choice can greatly influence the effectiveness and efficiency of the policy in delivering desired outcomes.

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Appendix

Proof of Proposition 3.1 . With commitment, the planner can commit to the policy rule in setting the instrument: $\tau = \mu\theta + \varphi k$. The representative individual agent's belief and behavior are determined taken the rule as given,

$$k = E[\tau] = \mu E[\theta] + \varphi E[k].$$

Noting that $E[k] = k$ (the representative agent knows his own action) and $E[\theta] = \theta$ (the representative agent knows fundamental), it implies that, in the REE, the aggregate action satisfies,

$$k = \frac{\mu}{1-\varphi}\theta.$$

The equilibrium instrument and aggregate outcome are given by:

$$\tau = \frac{\mu}{1-\varphi}\theta, \quad y = (1-\alpha)(\mu\theta + \varphi k) + \alpha k = \frac{\mu}{1-\varphi}\theta.$$

The welfare loss can be rewritten into:

$$\left[\chi\left(\frac{\mu}{1-\varphi} - \delta\right)^2 + (1-\chi)\left(\frac{\mu}{1-\varphi} - 1\right)^2\right]\theta^2.$$

It turns out that μ and φ show up in a composite bundle $g(\mu, \varphi) = \frac{\mu}{1-\varphi}$ in the loss function. The first order condition with respect to the rule $g(\mu, \varphi)$ leads to the following condition,

$$g(\mu, \varphi) = \chi\delta + 1 - \chi.$$

This condition shows that only the composite $g(\mu, \varphi)$ matters in Ramsey policy, whereas the particular values of μ and φ are irrelevant. In particular, one can see that it is also optimal for the instrument to only respond to the fundamental: $\varphi = 0$.

When Ramsey policy is implemented, the welfare loss is given by

$$\left[\chi(\chi\delta + 1 - \chi - \delta)^2 + (1-\chi)(\chi\delta + 1 - \chi - 1)^2\right]\theta^2.$$

The welfare loss reduces to zero only when $\delta = 1$ and stays positive when $\delta \neq 1$.

Proof to Proposition 3.2. With the discretionary policy, the policy maker perceives that the first stage of the policy game has forgone and formulates the policy taking k as given.

$$\tau = \mu^D \theta + \varphi^D k,$$

where μ^D and φ^D parameterize how the discretionary policy responds to the fundamental and the aggregate action respectively. The loss function can be rewritten into the following equation as the policy maker takes k as given,

$$\mathcal{L} = \chi \left((1-\alpha)\,\tau + \alpha k - \delta\theta \right)^2 + (1-\chi)\,(\tau - \theta)^2 .$$

The first order condition with respect to τ is given by

$$\chi\,(1-\alpha)\left((1-\alpha)\,\tau + \alpha k - \delta\theta\right) + (1-\chi)\,(\tau-\theta) = 0.$$

This implies the discretionary policy has to meet the following condition:

$$\mu^D = \frac{\chi\,(1-\alpha)\,\delta + (1-\chi)}{\chi\,(1-\alpha)^2 + (1-\chi)}, \quad \varphi^D = -\frac{\chi\alpha\,(1-\alpha)}{\chi\,(1-\alpha)^2 + (1-\chi)}.$$

It is obvious that $\varphi^D < 0$ when $\alpha \in (0,1)$ and $\varphi^D > 0$ when $\alpha < 0$.

Proof to Proposition 3.3. Noting that $E[k] = k$ (the representative agent knows his own action) and $k = E[\tau]$, it implies that, in any REE,

$$y = (1-\alpha)\,\tau + \alpha E[\tau].$$

With discretionary policy, we have $\tau = \mu^D\theta + \varphi^D k$, and $E[\tau] = \mu^D E[\theta] + \varphi^D E[k] = \mu^D\theta + \varphi^D k = \tau$. In the equilibrium $\tau = y = \frac{\mu^D}{1-\varphi^D}\theta$.

When the policy maker commits to the first stage announcement and sets τ to be the Ramsey policy, the corresponding loss function can be simplified into:

$$\mathcal{L}^R = \chi\,(1-\chi)\,(1-\delta)^2\,\theta^2.$$

On the other hand, when the policy maker acts at discretion, the loss function becomes:

$$\mathcal{L}^D = \frac{\chi(1-\chi)[a+b]^2}{d[c+d]^2}\theta^2,$$

where the parameters are given by

$$a = 2\alpha^2\chi - \alpha\chi - \alpha^3\chi + \alpha\chi\delta - \alpha^2\chi\delta = \alpha\chi(1-\alpha)(\alpha+\delta-1),$$

$$b = \delta + 3\alpha\chi - 3\alpha^2\chi + \alpha^3\chi - 3\alpha\chi\delta + 2\alpha^2\chi\delta - 1,$$

$$c = \chi\alpha(1-\alpha) > 0,$$

$$d = \chi\alpha^2 - 2\chi\alpha + 1 > 0.$$

In order to show that the Ramsey policy is superior in term of the social welfare compared with the discretionary policy, we need to show that $\mathcal{L}^R < \mathcal{L}^D$ always holds.

$$\mathcal{L}^R < \mathcal{L}^D \iff \chi(1-\chi)(1-\delta)^2\theta^2 < \frac{\chi(1-\chi)[a+b]^2}{d[c+d]^2}\theta^2.$$

This is equivalent to proving

$$(a+b)^2 > (\delta-1)^2(c+d)^2 d \iff (\delta-1)^2 d^2 > (\delta-1)^2(c+d)^2 d.$$

After eliminating $(\delta-1)^2 d$ from both sides of the inequality, we get

$$d > (c+d)^2 \iff \chi\alpha^2 - 2\chi\alpha + 1 > (1-\alpha\chi)^2 \iff 1 > \chi.$$

Since in our context, χ parameterizes the relative importance of the outcome's deviation from its target in the loss function, it must be smaller than one. Thus we have proved that $\mathcal{L}^R < \mathcal{L}^D$ must be true.

Proof to Proposition 4.1. Due to higher order uncertainty, agents underestimate the response of others

$$E_i[k] = \lambda k.$$

With the Ramsey policy, $\tau = \mu\theta + \varphi k$. This implies that the individual best response is

$$k_i = \mu E_i[\theta] + \varphi E_i[k] = \mu\theta + \varphi\lambda k.$$

It follows that the aggregate action is given by

$$k = \mu\theta + \varphi\lambda k = \frac{\mu}{1 - \varphi\lambda}\theta.$$

And the Ramsey policy is given by

$$\tau = \frac{1 + \varphi(1 - \lambda)}{1 - \varphi\lambda}\mu\theta = [1 + \varphi(1 - \lambda)]\, k.$$

The realized aggregate action is given by

$$k = \frac{1}{1 + \varphi(1 - \lambda)}\tau.$$

Then the outcome is given by

$$y = \left(1 - \alpha + \frac{\alpha}{1 + \varphi(1 - \lambda)}\right)\tau.$$

And it implies the implementability constraint is given by

$$\phi(\lambda) = 1 - \alpha + \frac{\alpha}{1 + \varphi(1 - \lambda)}.$$

Since with commitment, the policy maker can commit to any pair of (μ, φ), they can commit to φ such that

$$\phi(\lambda) = 1 - \alpha + \frac{\alpha}{1 + \varphi^*(1 - \lambda)} = \delta.$$

As a result,

$$\varphi = \frac{\delta - 1}{(1 - \alpha - \delta)(1 - \lambda)}.$$

Proof to Proposition 4.2. The agent's behavior is given by:

$$k_i = E_i[\tau].$$

Due to higher order doubts, agents under-estimate the response of others

$$E_i[k] = \lambda k.$$

With the discretionary policy, $\tau = \mu^D \theta + \varphi^D k$. This implies that the individual best response is given by

$$k_i = \mu^D E_i[\theta] + \varphi^D E_i[k] = \mu^D \theta + \varphi^D \lambda k.$$

It follows that the aggregate action is given by

$$k = \mu^D \theta + \varphi^D \lambda k = \frac{\mu^D}{1 - \varphi^D \lambda} \theta$$

And the discretionary policy is given by

$$\tau = \frac{1 + \varphi^D (1 - \lambda)}{1 - \varphi^D \lambda} \mu^D \theta = \left[1 + \varphi^D (1 - \lambda)\right] k.$$

The realized aggregate action is given by

$$k = \frac{1}{1 + \varphi^D (1 - \lambda)} \tau.$$

Then the outcome is given by

$$y = \left(1 - \alpha + \frac{\alpha}{1 + \varphi^D (1 - \lambda)}\right) \tau.$$

And it implies the implementability constraint is given by

$$\xi(\lambda) = 1 - \alpha + \frac{\alpha}{1 + \varphi^D (1 - \lambda)} = \frac{1 + (1 - \alpha) \varphi^D (1 - \lambda)}{1 + \varphi (1 - \lambda)} = 1 - \frac{\alpha \varphi^D (1 - \lambda)}{1 + \varphi^D (1 - \lambda)}.$$

Since with the discretionary policy, the optimality condition suggests that

$$\varphi^D = -\frac{\chi\alpha(1-\alpha)}{\chi(1-\alpha)^2 + (1-\chi)}.$$

And it implies that $\alpha\varphi^D < 0$. As a result,

$$\xi(\lambda) > 1 \text{ for } \lambda < 1, \quad \xi(1) = 1.$$

Taking the first order derivative of $\xi(\lambda)$ with respect to λ yields:

$$\frac{\partial \xi(\lambda)}{\partial \lambda} = \frac{\alpha\varphi^D}{[1-\varphi(1-\lambda)]^2} < 0.$$

As a result, $\xi(\lambda)$ is decreasing in λ.

Proof to Proposition 4.3. The loss function under discretionary policy is denoted as $\mathcal{L}^D$:

$$\mathcal{L}^D = \frac{\chi(1-\chi)[a\lambda + b]^2}{d[c\lambda + d]^2}\theta^2.$$

Specifically, with common knowledge, the social loss under discretionary policy is denoted as $\mathcal{L}^D_{CK}$:

$$\mathcal{L}^D_{CK} = \frac{\chi(1-\chi)[a + b]^2}{d[c + d]^2}\theta^2.$$

Denote $\phi(\lambda)$,

$$\phi(\lambda) = \frac{a\lambda + b}{c\lambda + d} = e - \frac{f/c}{\lambda + d/c}.$$

where the parameters a, b, c, d, e, f, g, h are defined as the functions of deep parameters α, δ, χ.

$$c = \chi\alpha(1-\alpha), \quad d = \chi\alpha^2 - 2\chi\alpha + 1 = \chi(1-\alpha)^2 + 1 - \chi, \quad e = \alpha + \delta - 1,$$
$$a = ce, \quad b = (\delta - 1)d - a, \quad f = \alpha[1 - \chi + \delta\chi(1-\alpha)].$$

It turns out that $\frac{f}{c} = \frac{1-\chi+\delta\chi(1-\alpha)}{\chi(1-\alpha)} > 0$ as we assumed $1 - \alpha > 0$ in this framework. Specifically $\phi(1) = (\delta - 1)\frac{1-\chi+\chi(1-\alpha)^2}{1-\chi+\chi(1-\alpha)}$, and $\phi(0) = \delta - 1 - \frac{\alpha\chi(1-\alpha)(\alpha+\delta-1)}{1-\chi+\chi(1-\alpha)^2}$.

Suppose $\alpha > 0$, then we have $\frac{d}{c} = \frac{\chi(1-\alpha)^2+1-\chi}{\chi\alpha(1-\alpha)} > 0$. Suppose $\alpha < 0$, then we have $\frac{d}{c} < -1$. Then $\phi(\lambda)$ is monotonically increasing for $\lambda \in [0, 1]$ no matter α is positive or negative.

1. For the case $\delta > 1$, $\phi(1) > 0$.
   (a) If $\phi(0) > 0$, or equivalently $\delta - 1 > \frac{ce}{d}$, then we have $0 < \phi(\lambda) < \phi(1), \forall \lambda \in [0, 1)$ and automatically $\mathcal{L}^D < \mathcal{L}^D_{CK}$.
   (b) If $\phi(0) < 0$ and $\phi(0) + \phi(1) > 0$, or equivalently $\frac{\chi\alpha^2(1-\alpha)}{2d^2-c^2} < \delta - 1 < \frac{ce}{d}$, then we have $|\phi(\lambda)| < \phi(1), \forall \lambda \in [0, 1)$ and $\mathcal{L}^D < \mathcal{L}^D_{CK}$.
   But if $\phi(0) < 0$ and $\phi(0) + \phi(1) < 0$, that is $\delta - 1 < \frac{\chi\alpha^2(1-\alpha)}{2d^2-c^2}$, then there exists $\lambda < \lambda^0$ such that $\mathcal{L}_D > \mathcal{L}^{CK}_D$, where λ^0 is determined by $\phi(\lambda^0) = -\phi(1)$.
2. For the case $\delta < 1$, $\phi(1) < 0$. Then we have $\phi(\lambda) < \phi(1) < 0, \forall \lambda \in [0, 1)$. Which implies that $\left[\phi(\lambda)\right]^2 > \left[\phi(1)\right]^2$, and $\mathcal{L}^D > \mathcal{L}^D_{CK}$.

Proof to Proposition 4.4. The loss function under restricted Ramsey policy and discretionary policy are respectively $\mathcal{L}^{RR}$ and $\mathcal{L}^D$:

$$\mathcal{L}^{RR} = \chi(1-\chi)(1-\delta)^2, \quad \mathcal{L}^D = \frac{\chi(1-\chi)[a\lambda+b]^2}{d[c\lambda+d]^2}\theta^2.$$

And denote $\phi(\lambda)$

$$\phi(\lambda) = \frac{a\lambda+b}{c\lambda+d} = e - \frac{f}{c\lambda+d}.$$

The claim that $\mathcal{L}^D$ is less than $\mathcal{L}^{RR}$ is equivalent to the following inequality:

$$\frac{[a\lambda+b]^2}{d[c\lambda+d]^2} < (1-\delta)^2.$$

Suppose $\delta > 1$, then the following condition should satisfy,

$$(1-\delta)\sqrt{d} < \phi(\lambda) < (\delta-1)\sqrt{d}.$$

Otherwise $\delta < 1$. then instead the following condition satisfies,

$$(\delta-1)\sqrt{d} < \phi(\lambda) < (1-\delta)\sqrt{d}.$$

The necessary condition for the existence of λ region such that discretionary policy outperforms the restricted Ramsey policy is that

$$\delta - 1 > 0.$$

Then we have

$$\phi(1) > (\delta - 1)\sqrt{d}.$$

For the λ region where the discretionary policy with higher doubts can outperform the restricted Ramsey policy, the following condition also has to satisfy,

$$\phi(0) < (\delta - 1)\sqrt{d}.$$

The inequality condition is equivalent to:

$$\frac{b}{d} < (\delta - 1)\sqrt{d} \iff (\delta - 1)\left(1 - \sqrt{d}\right) < \frac{a}{d} \iff (\delta - 1)\left(1 - \sqrt{d}\right) < -(\alpha + \delta - 1)\varphi.$$

Or equivalently,

$$0 < \delta - 1 < -\frac{\alpha\varphi}{1 + \varphi - \Delta}.$$

where

$$\Delta = \sqrt{\chi(1-\alpha)^2 + 1 - \chi}.$$

We have proven that when $\lambda = 1$, the discretionary policy is strictly inferior to the Ramsey policy, that is the following conditions hold.

$$\begin{aligned} &\texttt{if}\ \ \delta - 1 > 0 \implies \phi(1) > 0 \implies \phi(1) > (\delta - 1)\sqrt{d} \\ &\texttt{if}\ \ \delta - 1 < 0 \implies \phi(1) < 0 \implies \phi(1) < (\delta - 1)\sqrt{d} \end{aligned}$$

To ensure that $\mathcal{L}^D = 0$ is attainable, the following two conditions must be satisfied simultaneously,

$$\phi(1) > (\delta - 1)\sqrt{d} \implies \delta - 1 > 0,$$

$$\phi(0) < 0 \iff \frac{b}{d} < 0 \iff \delta - 1 < \frac{a}{d} = -\varphi(\alpha + \delta - 1) \implies \delta - 1 < -\frac{\alpha\varphi}{1 + \varphi}.$$

The first best can be reached only when $\phi\left(\lambda^{FB}\right) = 0$, which implies that $\lambda^{FB} = 1 + \frac{\delta - 1}{\varphi(\alpha + \delta - 1)}$.

Proof to Corollary 4.1. The loss function under discretionary policy is denoted as $\mathcal{L}^D$:

$$\mathcal{L}^D = \frac{\chi(1-\chi)[a\lambda + b]^2}{d[c\lambda + d]^2}\theta^2$$

denote $\phi(\lambda)$

$$\phi(\lambda) = \frac{a\lambda + b}{c\lambda + d} = e - \frac{f/c}{\lambda + d/c}$$

As λ decreases from 1 to 0, $\phi(\lambda)$ monotonically decreases (as shown in Proposition 4.3). Furthermore, the implementability constraint is characterized by $\xi(\lambda)$ (as shown in Proposition 4.2),

$$\xi(\lambda) = 1 - \frac{\alpha\varphi^D(1-\lambda)}{1 + \varphi^D(1-\lambda)}.$$

1. Consider the case where $0 < \delta - 1 < -\alpha\frac{\varphi^D}{1+\varphi^D}$. It is equivalent to condition $\phi(1) > 0$ and $\phi(0) < 0$. As a result, $\mathcal{L}^D$ first decreases to reach zero and then increases as λ decreases.
2. Consider the case where $\delta - 1 > -\alpha\frac{\varphi^D}{1+\varphi^D}$. This is equivalent to condition $0 < \phi(0) < \phi(1)$. As a result, $\mathcal{L}^D$ is monotonically increasing in λ.
3. Consider the case where $\delta - 1 < 0$, then we have $\phi(0) < \phi(1) < 0$. As a result, $\mathcal{L}^D$ is monotonically decreasing in λ.