Self-fulfilling Business Cycles with Production Network*

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Abstract

What is the role of production networks in inducing self-fulfilling business cycles? To answer this question, we build a multisector business cycle model with input-output linkages and credit constraints. Theoretically, we show that a single aggregate financial multiplier is sufficient to characterize the equilibrium determinacy, which hinges on the production network structure. Particularly, tightening credit constraints in upstream sector is more likely to generate self-fulfilling equilibria. Quantitatively, we evaluate the likelihood of indeterminate equilibria in the US from 1998 to 2020, and we find the economy is more susceptible to self-fulfilling fluctuations after the Great Recession.

Keywords: Endogenous Cycles, Indeterminacy, Production Networks, Credit Constraints, Financial Contagion.

JEL Classification: D24, E23, E32, E44.

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1. Introduction

Since the Great Recession, the role of financial frictions in shaping business cycle fluctuations has become a central theme in economics. It is well known that financial shocks themselves can be important driving forces (see, among others, Jermann and Quadrini (2012), Eggertsson and Krugman (2012), and Guerrieri and Lorenzoni (2017)). More surprisingly, severer financial frictions may also give rise to self-fulfilling business cycles (Liu and Wang (2014)). On the other hand, a rapidly growing literature on production networks (Gabaix (2011), Acemoglu, Akcigit, and Kerr (2016), Baqaee and Farhi (2020)) has pointed out that sectoral heterogeneities and input-output linkages are important in propagating local shocks. The goal of this paper is to understand the effects of the joint presence of credit constraints and production networks in inducing self-fulfilling business cycles.

We build a continuous-time multisector business cycle model with input-output linkages and credit constraints. In each production sector, a continuum of firms has access to a constant-return-to-scale technology that uses other sectors' outputs as intermediate goods, but they possess heterogeneous productivities and are subject to working capital constraints and fixed operational costs. In the absence of credit constraints, only the most productive firms operate. When such constraints are present, all firms with productivities higher than a threshold operate. Crucially, the credit constraint depends on the aggregate economic conditions in an endogenous way: an increase in economy-wide TFP not only directly improves production efficiency but also increases firms' equity value, which in turn relaxes their credit constraints. What follows is a reduction of misallocation: the cutoff productivity increases, resources shift towards more productive firms, and misallocation is mitigated. This indirect channel can be viewed as a particular financial multiplier. In addition, when firms are interconnected across sectors, an expansion in one sector boosts the demand for goods from other sectors, which in turn relaxes the credit constraints in other sectors. This interaction between the general equilibrium feedback effects due to trade linkages and the financial multiplier caused by credit constraints can induce sufficient amplification that makes the aggregate production function appear to display increasing returns to scale.

Our first result is an exact analytical characterization of the aggregate financial multiplier with dynamic linkage across sectors. We show that the rise of self-fulfilling business cycles hinges on the size of aggregate financial multiplier. Holding primary input factors fixed, the direct effect on aggregate output of a TFP shock in one sector equals its cost-based Domar weight, which is generically larger than the revenue-based Domar weight. These two Domar weights coincide only if the financial friction vanishes, in which case the familiar Hulten's theorem applies. Indirectly, the allocation efficiency in all sectors is improved. The indirect effects are governed by the magnitude of the fixed costs and productivity dispersion in each sector. The aggregate financial multiplier therefore also nests the input-output multiplier, which is greater than one.

Our second result involves the weight and the architecture of the production network and reveals their impact on self-fulfilling equilibrium. We show that the financial multiplier is non-monotonic in both weight and the architecture of the production network. First, consider a case in which all sectors are symmetric. The financial multiplier displays a U-shape in intermediate input share. The rise of intermediate input share has two effects: a "size effect" and a "diluting effect". The former effect indicates that the sales of intermediate firms increase, and this tends to amplify the financial multiplier as intermediate firms that are subject to the financial constraint expand. The latter effect occurs because as sales expand, fixed operating costs become relatively less important, which dampens the financial multiplier. The economy can be pushed into self-fulfilling cycles when either effect is strong enough. On the other hand, we can vary the structure of the production network. The aggregate financial multiplier again displays a U-shape in interconnection between sectors. We find that the network structure has an important but ambiguous effect in inducing a self-fulfilling equilibrium, in the sense that making sectors more or less interconnected can either increase or decrease the likelihood of economic indeterminacy.

Our third result demonstrates that tightening sector-specific credit constraints has heterogeneous effects on inducing self-fulfilling business cycles depending on a sector's position in the network. In general, tightening a sector's credit constraint leads to a larger aggregate financial multiplier and hence a higher chance of self-fulfilling business cycles. However, the strength of this channel depends on the relative importance of this sector in the economy. An upstream sector is more "critical", and the corresponding increase in the aggregate financial multiplier is more dramatic there. It follows that tightening the financial constraints in these more critical sectors would more easily induce self-fulfilling business cycles.

Quantitatively, we evaluate the likelihood of self-fulfilling business cycles in the US from 1998 to 2020. We show that the aggregate financial multiplier can be used as a sufficient statistic to describe the endogenous risk in the economy: the sunspot fluctuations are more likely to occur when it falls into indeterminacy region. During the 2008 financial crisis, the aggregate financial multiplier is relatively high and drives the economy to be more exposed to the sunspot risks.

Overall, our results highlight how the nature of interactions between the input-output linkages and credit constraints shapes the aggregate self-fulfilling fluctuations.

Related Literature. To the best of our knowledge, our paper is the first to shed light on the role of production networks in shaping self-fulfilling business cycles. Correspondingly, our paper most directly draws from and contributes to the literature on self-fulfilling equilibria in real business cycles. Benhabib and Farmer (1994) first notes that increasing returns to scale can generate indeterminate equilibria. Farmer and Guo (1994), Basu and Fernald (1995), and Basu and Fernald (1997) examine the quantitative importance of such indeterminacy. To generate empirically plausible increasing returns, Galí (1993), Schmitt-Grohé (1997) and Wang and Wen (2008) resort to countercyclical markup, Benhabib and Farmer (1996) and Benhabib and Nishimura (2012) turn to two-sector model with mild within-sector externality, while Wen (1998) and Benhabib and Wen (2004) introduce variable capacity into the discussion. We contribute to this research agenda by introducing a production network that can endogenously amplify increasing returns. Our paper also complements to the literature on expectation-driven self-fulfilling business cycles. A partial list includes Benhabib, Wang, and Wen (2015), Chahrour and Gaballo (2017), and Acharya, Benhabib, and Huo (2021).

Our paper also belongs to the active recent research agenda on production networks. Following the pioneering contribution of Long and Plosser (1983) on multisector real business cycles, Foerster, Sarte, and Watson (2011), Gabaix (2011), Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012), Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), Acemoglu, Akcigit, and Kerr (2016), Atalay (2017), Oberfield (2018), Baqaee (2018), Baqaee and Farhi (2018), Baqaee and Farhi (2019), Liu (2019), Altinoglu (2020), Luo (2020), Bigio and La'o (2020), and Carvalho, Nirei, Saito, and Tahbaz-Salehi (2021), among others, enrich the theory and provide econometric evidence. Additionally, see Carvalho and Tahbaz-Salehi (2019) for a comprehensive survey on production networks. We apply the insights and tools developed by this body of work which mainly focuses on static models to the possibility of indeterminate equilibria in RBC models.

The most related paper to ours is Liu and Wang (2014), which shows that financial frictions not only amplify the cycles but can also generate self-fulfilling business cycles due to an endogenous aggregation with increasing returns to scale. There are mainly three key differences between Liu and Wang (2014) and our paper. First, the model in Liu and Wang (2014) considers production economy without intermediate goods input, while our paper considers a more complicated production network, treating Liu and Wang (2014) as a special case. That being said, the second most innovative part of our paper is that we investigate the implications of the network structure in a multisector model and the input-output structure is shown to shift the indeterminate region, i.e., the possibility region of endogenous cycles. Lastly, since our model builds a richer production structure which allows for heterogeneous sectoral financial constraints, we are able to identify the key sector whose financial condition is crucial for the stability of equilibrium.

The remainder of our paper is organized as follows. Section 2 introduces our basic theoretical model. Section 3 presents our investigation of aggregate financial multiplier and self-fulfilling business cycles when altering weight and architecture of the production network. Section 4 evaluates the importance of sectors at different positions in the network in shaping self-fulfilling business cycles. Section 5 quantitatively evaluates the likelihood of self-fulfilling business cycles. Section 6 concludes the paper, while the Appendix contains all the proofs.

2. Model

Time is continuous and indefinite. The economy is populated by a continuum of homogeneous households, a continuum of final-goods producers and N continua of intermediate goods producers. All intermediate goods producers in same sector are solely owned by one entrepreneur who is labeled by $i \in \{1, \ldots, N\}$.

2.1 Households

There is no intertemporal borrowing and lending for households or entrepreneurs. The only way to transfer wealth over time is by accumulating physical capital.[1] We model the worker side via a representative household with preferences given by

$$\max_{C_{h,t}, L_t, I_{h,t}} \mathbb{E}_0 \int_0^\infty e^{-\rho_h t} \left[u(C_{h,t}) - h(L_t) \right] dt, \tag{2.1}$$

where ρ_h is the discount factor, $C_{h,t}$ is the consumption rate, L_t is the labor supply, and $I_{h,t}$ is the capital investment rate. We also impose that $u(X) = \log X$ and $h(X) = \psi \frac{X^{1+\gamma}}{1+\gamma}$, where ψ is the disutility from working, and γ is the inverse Frisch elasticity. The budget constraint of the representative household is

$$C_{h,t} + I_{h,t} \leq R_t K_{h,t} + W_t L_t, \tag{2.2}$$

where R_t and W_t denote the capital rental price and wage, respectively. $K_{h,t}$ is capital owned by the household. The household also has access to a linear technology to transform final goods into investment goods $I_{h,t}$. The law of motion of capital $K_{h,t}$ is

$$\dot{K}_{h,t} = -\delta K_{h,t} + I_{h,t}.$$

The household takes R_t and W_t as given and chooses a path of consumption rate, working intensity and investment rate, $C_{h,t}$, L_t and $I_{h,t}$, to maximize their utility function (2.1).

2.2 Final and Intermediate Goods Producers

The final good producers simply assemble intermediate goods and have no access to any saving technology.[2] The price of final good is normalized to be 1. A representative final goods producer takes intermediate goods prices and the final good (numéraire) price as given and chooses the amount of

[1]This assumption is introduced to emphasize the impact of financial frictions on capital misallocation.
[2]There is no intertemporal choice for the final goods producers whatsoever.

intermediate goods input to maximize its per period profit,

$$\max_{\{X_{it}\}} \left\{ Y_t - \sum_{i=1}^{N} P_{it} X_{it} \right\}. \tag{2.3}$$

X_{it} denotes the input of sector i's intermediate goods, and P_{it} is the price for such intermediate goods. The production function for final goods Y_t takes Cobb-Douglas form,

$$Y_t = \prod_{i=1}^{N} X_{it}^{\varphi_i},$$

where we normalize the final goods production to be constant returns to scale, i.e., $\sum_{i=1}^{N} \varphi_i = 1$ with all $\varphi_i > 0$.

Each entrepreneur owns a sector, and each sector is denoted by $i \in \{1, ..., N\}$. There exists a continuum of firms in each sector, and each firm is indexed by $\iota \in [0, 1]$. Entrepreneurs have preferences given by

$$\max_{C_{e,i}, I_{e,i}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho_e t} \left[u(C_{e,it}) \right] dt, \tag{2.4}$$

where ρ_e is the discount rate for the entrepreneurs and $C_{e,it}$ is their consumption rate. The budget constraints for the entrepreneurs are

$$C_{e,it} + I_{e,it} \leq D_{e,it} + R_t K_{e,it}. \tag{2.5}$$

In the spirit of Kiyotaki and Moore (1997) and Liu and Wang (2014), we assume that $\frac{\rho_h}{\rho_e}$ is small enough that entrepreneurs have no incentive to accumulate any capital in equilibrium, i.e., $I_{e,it} = 0, K_{e,it} = 0$. Moreover, $D_{e,it}$ denotes the dividends received by the entrepreneurs, and $\Pi_{it}(\iota)$ is the profit of firm ι operating in sector i. Since all firms in sector i are solely owned by one entrepreneur, the dividends come from the total profits of all firms,

$$D_{e,it} = \int_0^1 \Pi_{it}(\iota) d\iota. \tag{2.6}$$

Firms also have to pay a fixed operating cost Φ_i in order to stay in business, and this fixed cost is prepaid to draw an individual productivity shock. If the firm stays inactive for some period (because of a low productivity draw) but remains in the business in the hope of becoming profitable later, the fixed cost Φ_i must nevertheless be paid. The fixed cost is financed by issuing equity to the entrepreneur who owns this sector.[3]

[3]Note here that by law of large numbers, the firms making money still outweigh the dormant firms. Therefore, entrepreneurs always enjoy a positive amount of consumption.

Production Technology. Each firm ι in sector i has access to a constant returns to scale technology that transforms capital, labor and intermediate goods into sector i goods,

$$O_{it}(\iota) = A_{it} Z_{it}(\iota) \left[K_{it}(\iota)^{\varkappa_i} L_{it}(\iota)^{1-\varkappa_i} \right]^{1-\alpha_{M,i}} \left[\prod_{j=1}^{N} S_{ijt}(\iota)^{\omega_{ij}} \right]^{\alpha_{M,i}},$$

where $O_{it}(\iota)$ is the firm's output and P_{it} is the price of this product. A_{it} is the sector-specific productivity shock. An operating firm rents capital $K_{it}(\iota)$ from households at rental rate R_t, hires labor $L_{it}(\iota)$ at the competitive real wage W_t, and employs a bundle of material goods $\prod_{j=1}^{N} S_{ijt}(\iota)^{\omega_{ij}}$. Specifically, $S_{ijt}(\iota)$ denotes the intermediate input, which is produced by sector j and used by sector i at time t, and its price is P_{jt}. $\alpha_{M,i}$ is the intermediate input share, out of which ω_{ij} is the j sector intermediate input share used by sector i. For the remaining expenditure, κ_i is used on renting capital.

Finally, $Z_{it}(\iota)$ is the firm-specific productivity shock, assumed to be i.i.d. both over time and across firms. $F_i(\cdot)$ is the cumulative density function of $Z_{it}(\iota)$ in sector i. We normalize $\mathbb{E}_i[Z_{it}(\iota)] = 1$ for all sectors. To obtain a sharp result, throughout the paper, we let $Z_i(\iota)$ conform to a Pareto distribution to obtain a closed-form solution, i.e., $F_i(Z_{it}) = 1 - \left(Z_{it}/\underline{Z}_i\right)^{-\eta_i}$ with $\underline{Z}_i = 1 - 1/\eta_i < 1$.[4]

We can also denote $\alpha_{K,i} = (1-\alpha_{M,i})\varkappa_i, \alpha_{L,i} = (1-\alpha_{M,i})(1-\varkappa_i), \alpha_{S,ij} = \alpha_{M,i}\omega_{ij}$; then, $\alpha_{K,i}, \alpha_{L,i}$ and $\alpha_{S,ij}$ represent the elasticities of output with respect to the capital input, labor input and intermediate goods input, respectively. Since we assume constant returns to scale for firm production, $\alpha_{K,i} + \alpha_{L,i} + \sum_{j=1}^{N} \alpha_{S,ij} = 1$. As a result, the production function is equivalent to

$$O_{it}(\iota) = A_{it} Z_{it}(\iota) K_{it}(\iota)^{\alpha_{K,i}} L_{it}(\iota)^{\alpha_{L,i}} \prod_{j=1}^{N} S_{ijt}(\iota)^{\alpha_{S,ij}}.$$

Working Capital Constraint. Firms do not make upfront payments to households and to other firms. Instead, all the working capital is loaned to the firms as credit. Since firms have limited liability, the credit offered is tailored to provide a sufficient repayment incentive. This working capital loan is made before firms draw their idiosyncratic productivities, so creditors must form expectations about firms' value before the realization of the productivity shocks. We show in appendix A. 3 that the IC condition takes a very simple form:

$$R_t K_{it}(\iota) + W_t L_{it}(\iota) + \sum_{j=1}^{N} P_{jt} S_{ijt}(\iota) \leq \Theta_i V_{it} \equiv \overline{B}_{it}. \tag{2.7}$$

V_{it} is an individual firm's expected value. Effectively, $\Theta_i \in [0,1]$ represents the degree of efficiency of the credit markets: $\Theta_i = 1$ corresponds to a credit market that offers borrowers maximal amount of

[4]Mathematical requirement: $\eta_i > 2$. Consequently, $\mathbb{E}[Z_{it}(\iota)] = 1$, $Var[Z_{it}(\iota)] = \frac{1}{\eta_i(\eta_i-2)}$

loan that can be guaranteed to be repaid, and $\Theta_i = 0$ corresponds to the case where the credit market is completely shut down. We use $\overline{B}_{it}$ to denote the endogenous borrowing limit in sector i.

We show in appendix A. 4 that with the presence of the working capital constraint, there exists a cutoff productivity Z^*_{it} for each sector, above which the firm participates in production; otherwise, the firm stays inactive. Thus, only firms with sufficiently high productivities choose to operate, and all operating firms will borrow to the limit. Because firms within a sector produce homogeneous goods and the marginal revenues from producing such goods are the same for all firms, the same amounts of capital, labor and intermediate goods are hired by heterogeneous operating firms. However, firms with higher idiosyncratic productivities effectively enjoy a lower marginal cost of producing. As a result, firms will borrow to their limit whenever their marginal costs are lower than their marginal revenues, and such limit is regulated by a common sector-level credit constraint. However, since the productivities are different, firms with higher productivities will produce more even with the same amount of input.

2.3 Input-Output Linkages and Aggregation

In this section, we introduce input-output linkages between sectors. We define input-output matrices, Leontief inverse matrices and Domar weights in this economy. This section builds on concepts widely used in the literature on production networks (see, for example, Baqaee and Farhi (2018) and Baqaee and Farhi (2020)).

Input-output Matrices. We define the revenue-based input-output matrix to be $\widetilde{\boldsymbol{\alpha}}_{St}$ and the cost-based input-output matrix to be $\boldsymbol{\alpha}_{St}$. The ij-th element of $\widetilde{\boldsymbol{\alpha}}_{St}$ is sector i's expenditure on the intermediate goods from sector j as a share of i's total revenue, while the ij-th element of $\boldsymbol{\alpha}_{St}$ corresponds to the elasticity of output with respect to intermediate goods input from sector j.

$$\widetilde{\alpha}_{S,ijt} = \frac{P_{jt}S_{ijt}}{P_{it}O_{it}}, \qquad \alpha_{S,ijt} = \frac{P_{jt}S_{ijt}}{W_tL_{it} + R_tK_{it} + \sum_{j=1}^{N} P_{jt}S_{ijt}}.$$

Notice that there is a wedge between the sector-level endogenous expenditure shares and the primitive firm-level elasticities of output with respect to variable inputs. We assume the wedge $\mathbb{E}_{it}\left(Z_{it}(\iota)/Z^*_{it}|Z_{it} \geq Z^*_{it}\right)$ weakly decreases with Z^*_{it}.[5]

$$\widetilde{\alpha}_{S,ijt} = \alpha_{S,ij}\frac{Z^*_{it}}{\mathbb{E}_i\left(Z_{it}(\iota)|Z_{it}(\iota) \geq Z^*_{it}\right)}.$$

[5] $\mathbb{E}_i\left(Z_{it}(\iota)/Z^*_{it}|Z_{it}(\iota) \geq Z^*_{it}\right) \geq 1$ creates a wedge between the average and the lower bound of an operating firm's productivity. This assumption basically states that such a wedge weakly narrows when the productivity floor is raised. This is a very mild assumption because intuitively when the productivity floor increases, the dispersion of productivities decreases. Since we assume the Pareto distribution for idiosyncratic productivity shocks, we immediately have $\mathbb{E}_i\left(Z_{it}(\iota)/Z^*_{it}|Z_{it}(\iota) \geq Z^*_{it}\right) = 1/\underline{Z}_i$, which is a constant and independent of Z^*_{it} and the assumption is satisfied.

Analogously, we can define the cost-based capital share and labor share,

$$\widetilde{\alpha}_{L,it} = \alpha_{L,i} \frac{Z_{it}^{*}}{\mathbb{E}_i \left(Z_{it}(\iota) | Z_{it}(\iota) \geq Z_{it}^{*} \right)}, \quad \widetilde{\alpha}_{K,it} = \alpha_{K,i} \frac{Z_{it}^{*}}{\mathbb{E}_i \left(Z_{it}(\iota) | Z_{it}(\iota) \geq Z_{it}^{*} \right)}$$

Leontief Inverse Matrix. We define the revenue-based Leontief inverse matrix $\widetilde{\boldsymbol{\Psi}}_t$ and cost-based Leontief inverse matrix $\boldsymbol{\Psi}_t$ as

$$\widetilde{\boldsymbol{\Psi}}_t = \left(\boldsymbol{I} - \widetilde{\boldsymbol{\alpha}}'_{S,t} \right)^{-1} = \sum_{k=0}^{\infty} \left(\widetilde{\boldsymbol{\alpha}}'_{S,t} \right)^k, \qquad \boldsymbol{\Psi}_t = \left(\boldsymbol{I} - \boldsymbol{\alpha}'_{S,t} \right)^{-1} = \sum_{k=0}^{\infty} \left(\boldsymbol{\alpha}'_{S,t} \right)^k.$$

Intuitively, the ij-th element of $\widetilde{\boldsymbol{\Psi}}_t$ is a measure of i's total reliance on j as a supplier. And the ij-th element of $\boldsymbol{\Psi}_t$ records the direct and indirect exposures of the cost of i to the price of j through the production network. Note that this is still a partial-equilibrium elasticity where factor prices are considered fixed.

Domar Weights. We define the revenue-based Domar weight $\widetilde{\lambda}_{it}$ of the intermediate goods producer i as its sales share as a fraction of aggregate output

$$\widetilde{\lambda}_{it} = \frac{P_{it} O_{it}}{Y_t}.$$

In general, $\sum_{i=1}^{N} \widetilde{\lambda}_i > 1$, since there are not only final sales but also intermediate sales. Domar weights are a measure of sector weights when we integrate or aggregate all the sectors. Revenue-based Domar weight $\widetilde{\boldsymbol{\lambda}}_t$ is determined by Leontief inverse matrix and final good expenditure share. And we can analogously define the cost-based Domar weight $\boldsymbol{\lambda}_t$ to measure the importance of i as a supplier for final goods producers,

$$\widetilde{\boldsymbol{\lambda}}_t = \left(\boldsymbol{I} - \widetilde{\boldsymbol{\alpha}}'_{S,t} \right)^{-1} \boldsymbol{\varphi} = \widetilde{\boldsymbol{\Psi}}_t \boldsymbol{\varphi}, \qquad \boldsymbol{\lambda}_t = \left(\boldsymbol{I} - \boldsymbol{\alpha}'_{S,t} \right)^{-1} \boldsymbol{\varphi} = \boldsymbol{\Psi}_t \boldsymbol{\varphi}.$$

Sectoral Misallocation and Shock Amplification. Due to imperfect contract enforcement, productive firms cannot operate at their full capacities, and their production is restricted by working capital constraints. This leads to misallocation and the suppression of aggregate productivity. Apparently, without credit constraints, only firms with the highest productivities operate. However, when credit constraints take effect, less productive firms can also participate, and such misallocation lowers average productivity. As a consequence, with the presence of working capital constraints, an increase in sectoral output will have an additional effect of relaxing the borrowing limits, which in turn drives up the average productivity in this sector and results in yet higher sectoral output. Credit constraints create a positive feedback loop that amplifies productivity shocks. We refer to this feedback loop as "reallocation effect". We show how such reallocation effect manifests itself in the borrowing and lending process in the following proposition.

Proposition 2.1. *Given the vector of cutoff productivities* $\boldsymbol{Z}_t^* = \left[Z_{1t}^*, ..., Z_{Nt}^*\right]'$ *and aggregate output* Y_t*, the loan-to-output ratio depends on the tightness of the credit constraints and the profitability of the average firm,*

$$\frac{\overline{B}_{it}}{Y_t} = \frac{\Theta_i}{\rho_e}\left[\xi\left(Z_{it}^*\right) - \phi_{it}\right] \equiv g(Z_{it}^*), \tag{2.8}$$

where $\phi_{it} = \Phi_i / Y_t$. $\xi(Z_{it}^*) - \phi_{it}$ *is the expected marginal profit for a firm,*

$$\xi(Z_{it}^*) = \widetilde{\lambda}_{it}\left(1 - \widetilde{\alpha}_{K,it} - \widetilde{\alpha}_{L,it} - \sum_{j=1}^{N}\widetilde{\alpha}_{S,ijt}\right).$$

On the other hand, the credit demand monotonically maps the loan-to-output ratio to cutoff productivity,

$$\frac{\overline{B}_{it}}{Y_t} = \frac{\widetilde{\lambda}_{it} Z_{it}^*}{\int_{Z_{it}^*} Z_{it}(\iota) dF(Z_{it}(\iota))} \equiv h(Z_{it}^*). \tag{2.9}$$

In addition, $g(Z_{it}^*)$ *is decreasing in* Z_{it}^**, while* $h(Z_{it}^*)$ *is weakly increasing in* Z_{it}^**.*

Proposition 2.1 pins down the supply of credit (2.8) and the demand of credit (2.9). Given the aggregate output Y_t, the cutoff productivities can be uniquely determined,

$$\frac{\Theta_i}{\rho_e}\left[\xi\left(Z_{it}^*\right) - \phi_i\right] = \frac{\widetilde{\lambda}_{it} Z_{it}^*}{\int_{Z_{it}^*} Z_{it}(\iota) dF(Z_{it}(\iota))}.$$

The cutoff productivities can immediately be connected to aggregate output with Pareto distribution,

$$Z_{it}^* = \left[\frac{\Theta_i}{\rho_e(\eta_i - 1)}\left(1 - \frac{\eta_i \Phi_i}{\widetilde{\lambda}_i Y_t}\right)\right]^{1/\eta_i} \underline{Z}_i.$$

This result shows that Z_{it}^* is increasing in Θ_i. If contract enforcement is stronger (i.e., Θ_i is higher), more credit will be made available for the more productive firms, and the resources will shift towards those firms. In accordance, the cutoff productivity Z_{it}^* will be increased. By the same token, Z_{it}^* also strictly increases in Y_t, provided that the fixed cost is positive. Higher aggregate output inflates the value of productive firms, which in turn relaxes the credit constraints for more productive firms and facilitates resource reallocation to the more productive firms.

Our intuition regarding the self-fulfilling business cycles hinges on an endogenous amplification mechanism. To understand how credit constraints amplify business cycle fluctuations, let us consider a hypothetical increase in sector-specific productivity A_i and see how the labor market responds.

On the labor supply side, there exist three competing forces. First, the marginal productivity of labor increases, as does the wage. The wage effect will boost labor supply. Second, a higher marginal productivity of capital drives up the interest rate. The interest rate effect makes households more willing to work today. Third, the production expansion allows households to earn a higher income. The wealth effect makes households desire more leisure and supply less labor. The overall change in supply of labor is determined by the relative forces of these three effects. Under usual configuration, the labor supply curve moves up.

On the labor demand side, there are also three forces taking effect, each reinforcing the other. First, when the marginal productivity of labor increases, firms tend to hire more workers. Second, with the presence of credit constraints, production expansion leads to an increase in firm value and thus enables productive firms to borrow more. Besides, with credit constraints, the loan-to-output ratio responds more than proportionately to changes in total output, because higher output reduces the average fixed cost and thus alleviates the effective credit constraint. This reallocation effect drives up the sector-level endogenous productivity and allows firms to hire more workers. Third, an increase in sector i's productivity will also positively affect other sectors' production through the input-output linkages, which creates positive reallocation effects throughout the economy. We call this effect as "reallocation through network effects". But of course, a sector level productivity shock may have mixed effect on the total output depending on the architecture of the network and the position of the sector in the network. All three forces will shift the labor demand curve out.

The equilibrium employment is jointly determined by the supply and demand side of the labor market as discussed above. But no matter what direction the equilibrium employment eventually moves, the financial constraints and input-output linkages are more likely to amplify such movement.

Aggregation. We can map the multisector economy to the prototypical representative-firm economy. In proposition 2.2, we will derive a reduced-form aggregate production function. When we aggregate the whole economy, the final good production can be expressed as a Cobb-Douglas production function using aggregate capital and aggregate labor. The endogenous TFP turns to be the Domar-weighted geometric average of all sectors' endogenous productivities, corrected by the general equilibrium effect from the resource allocation. Meanwhile, the capital and labor share are endogenous as well, which in theory is time varying. Our aggregation results are complementary to the related literature such as Jones (2005), Lagos (2006), Moll (2014), Mangin (2017) and Baqaee and Farhi (2018), among others. The following representation shows how sectoral distortions manifest at the aggregate level.

Proposition 2.2. ***(Aggregate output and endogenous TFP)*** *The aggregate output is given by*[6]

$$Y_t = A_t K_t^{\underline{\alpha}_{K,t}} L_t^{\underline{\alpha}_{L,t}}, \tag{2.10}$$

where the aggregate elasticities of final good output with respect to capital and labor are given by $\underline{\alpha}_{K,t} = \boldsymbol{\lambda}_t' \boldsymbol{\alpha}_K$, $\underline{\alpha}_{L,t} = \boldsymbol{\lambda}_t' \boldsymbol{\alpha}_L$, *with* $\underline{\alpha}_{K,t} + \underline{\alpha}_{L,t} = 1$. *We define endogenous TFP as*

$$A_t = \zeta_t \prod_{i=1}^{N} \widetilde{A}_{it}^{\lambda_{it}},$$

where $\widetilde{A}_{it} = A_{it} \mathbb{E}_i \left(Z_{it} | Z_{it} \geq Z_{it}^* \right) \geq A_{it}$ *can be interpreted as endogenous sector-specific productivity. Moreover, we define* ζ_t *as the TFP component that is affected by the allocation of factors and intermediate goods,*

$$\zeta_t = \prod_{i=1}^{N} \left[k_{it}^{\alpha_{K,i}} l_{it}^{\alpha_{L,i}} \prod_{j=1}^{N} \left(\widetilde{\alpha}_{S,ijt} \frac{\widetilde{\lambda}_{it}}{\widetilde{\lambda}_{jt}} \right)^{\alpha_{S,ij}} \right]^{\lambda_{it}} \cdot \prod_{i=1}^{N} \left(\frac{\varphi_i}{\widetilde{\lambda}_{it}} \right)^{\varphi_i},$$

with $k_{it} = K_{it}/K_t$ *and* $l_{it} = L_{it}/L_t$ *being the capital and the labor share employed by each sector.*

Furthermore, the GDP can be obtained using the value-adding approach,

$$GDP = Y_t - \sum_{i=1}^{N} P_{it} X_{it} + \sum_{i=1}^{N} \left[P_{it} O_{it} - \sum_{j=1}^{N} P_{jt} S_{ijt} - \Phi_i \right] = Y_t - \sum_{i=1}^{N} \Phi_i.$$

Another observation, as shown in appendix A. 6, is that the accumulation of aggregate capital is

$$\dot{K}_t = -\delta K_t + \left(\overline{\alpha}_{K,t} + \overline{\alpha}_{L,t} \right) Y_t - C_{h,t}. \tag{2.11}$$

where $\overline{\alpha}_{L,t} \equiv \widetilde{\boldsymbol{\lambda}}_t' \widetilde{\boldsymbol{\alpha}}_{L,t}$ and $\overline{\alpha}_{K,t} \equiv \widetilde{\boldsymbol{\lambda}}_t' \widetilde{\boldsymbol{\alpha}}_{K,t}$ as shown in appendix A. 5, are the total expenditure shares of labor and capital. It follows that the accumulation of capital is related to aggregate output Y_t and household consumption $C_{h,t}$. Note that the households' decisions regarding labor supply L_t and consumption $C_{h,t}$ are all nonlinear functions of K_t. In addition, $(\overline{\alpha}_{K,t} + \overline{\alpha}_{L,t})$ could also be highly nonlinear in principle since Z_{it}^* is a complicated function of Y_t. Therefore, the law of motion of capital is highly nonlinear.

[6]If $N = 1$, i.e., for the one-sector model, then immediately $\zeta = 1$, and the aggregate output in equation (2.10) is simply reduced to

$$Y_t = A_t \mathbb{E} \left(Z_t | Z_t \geq Z_t^* \right) \cdot K_t^{\alpha_K} L_t^{\alpha_L},$$

This coincides with the one-sector model in Moll (2014), Liu and Wang (2014), where A_t is the aggregate TFP shock and α_K, α_L are the capital share and labor share.

2.4 Equilibrium Definition and Characterization

Market Clearing Conditions. The markets for capital, labor and intermediate goods all clear

$$K_{h,t} = \sum_{i=1}^{N} K_{it}, \quad L_t = \sum_{i=1}^{N} L_{it}, \quad O_{it} = X_{it} + \sum_{j=1}^{N} S_{jit}. \tag{2.12}$$

In addition, summing up entrepreneurs' and workers' budget constraints and by the aforementioned market clearing conditions, we can obtain the aggregate resource constraints in this economy,

$$Y_t - \sum_{i=1}^{N} \Phi_i = C_{h,t} + C_{e,t} + I_{h,t} + I_{e,t}. \tag{2.13}$$

Competitive Equilibrium. Given sequences of sector-level productivities A_{it} and distributions of firm-specific productivities $F(Z_{it})$, a competitive equilibrium consists of a set of prices $\{P_{it}, R_{it}, W_{it}\}$, the consumption, labor supply and investment decisions of the households $\{C_{h,t}, L_t, I_{h,t}\}$, the consumption of the entrepreneurs $\{C_{e,it}\}$, the intermediate goods input chosen by final good producers $\{X_{it}\}$, the capital, labor and intermediate goods input, and the borrowing amount chosen by the intermediate goods producers $\left\{K_{it}(\iota), L_{it}(\iota), S_{ijt}(\iota)\right\}$such that:

i. households maximize the utility (2.1) subject to their budget constraints (2.2);

ii. entrepreneurs maximize their utility (2.4) subject to their budget constraints (2.5);

iii. final good producers choose intermediate goods inputs to maximize their profits (2.3);

iv. intermediate goods producers choose capital, labor and intermediate goods inputs to maximize their profits (A.3) subject to the collateral constraint (A.4) as shown in appendix A. 3; and

v. the markets for all goods and factors clear by (2.12) and (2.13).

3. Multisector Business Cycles

We have established that the reallocation effect and production network are essential for amplifying productivity shocks. We now turn to studying self-fulfilling business cycles. We will show that credit constraints together with the production network can make the aggregate economy observationally equivalent to an economy that operates at increasing returns to scale, thus making it more likely to be exposed to self-fulfilling fluctuations.

3.1 Aggregate Financial Multiplier

In this section, we will convince the readers that whether an economy is exposed to self-fulfilling fluctuations can be summarized by a single characteristic variable: *aggregate financial multiplier* μ. Proposition 3.1 shows that equilibrium indeterminacy arises if the size of aggregate financial multiplier falls within certain range.

Proposition 3.1. ***(Equilibrium indeterminacy)*** *The necessary and sufficient condition for equilibrium indeterminacy is given by*

$$\max\left\{\mu_1^*, \mu_2^*\right\} < \mu < \mu_3^*,$$

where μ is the aggregate financial multiplier,

$$\mu \equiv \left(1 - \sum_{i=1}^{N} \upsilon_i \lambda_i\right)^{-1} = (1-\varrho)^{-1}.$$

Here, we write $\varrho = \sum_{i=1}^{N} \upsilon_i \lambda_i$, and $\upsilon_i = \frac{\Phi_i}{Y\widetilde{\lambda}_i - \eta_i \Phi_i} = \left(\frac{\widetilde{\lambda}_i Y}{\Phi_i} - \eta_i\right)^{-1}$.

$$\mu_1^* = \frac{1+\gamma}{\underline{\alpha}_L}, \quad \mu_2^* = \frac{\delta(1+\gamma)}{\frac{\overline{\alpha}_K + \overline{\alpha}_L}{\overline{\alpha}_K}(\delta+\rho_h)(1+\gamma)\underline{\alpha}_K - \underline{\alpha}_L \rho_h}, \quad \mu_3^* = \frac{1}{\underline{\alpha}_K}.$$

The indeterminacy region is defined by $\left(\max\left\{\mu_1^*, \mu_2^*\right\}, \mu_3^*\right)$. μ_1^* and μ_2^* jointly determine the lower bound of the indeterminacy region. Notice that indeterminacy will only arise when a hypothetical increase in expected future equity value can be validated in the equilibrium. With a reasonably high aggregate financial multiplier, the reallocation effect can generate sufficiently large increment in aggregate TFP. This makes the labor demand shift sufficiently up to offset the wealth effect on labor supply side. As a result, the equilibrium employment and the expected future equity value indeed increase as expected. On the other hand, μ_3^* defines the upper bound. If $\mu > \mu_3^*$, the amplification effect is too strong to the extend that it generates endogenous growth rather than endogenous cycle. Notice that both the size of aggregate financial multiplier and its range that creates indeterminacy are influenced by the production network.

In order to establish the equivalence shown in proposition 3.1, let us first study the local dynamics around steady state and show how we obtain the exact analytical characterization of aggregate financial multiplier μ.[7] We henceforth work with the log-linearized solution around a steady state. We denote the steady state of X_t as X and the percentage deviation of X_t from the steady state as $\widehat{X}_t$.

[7]We are also able to characterize the global dynamics for this economy, and the discussion is in Appendix C. 3.

Lemma 1. *The log-linearized system of output and sectoral cutoff productivity is given by*

$$\widehat{Z}^*_{it} = \upsilon_i \widehat{Y}_t, \tag{3.1}$$

$$\widehat{Y}_t = \sum_{i=1}^{N} \lambda_i \widehat{A}_{it} + \sum_{i=1}^{N} \lambda_i \widehat{Z}^*_{it} + \underline{\alpha}_K \widehat{K}_t + \underline{\alpha}_L \widehat{L}_t. \tag{3.2}$$

Lemma 1 highlights a positive feedback loop in local dynamics. Let us consider a hypothetical improvement in sector-specific productivity A_i. From equation (3.2), this will lead to an increase in total output Y_t. How does an increase in Y_t affect the sectoral productivity cutoff Z^*_{it}? The fact that $\upsilon_i > 0$[8] and equation (3.1) underscore that a higher aggregate output improves allocation efficiency in sector i. We call this reallocation effect: a higher total output helps alleviate the effective sectoral financial constraint and improves the sectoral productivity by reallocating credit to more productive firms. Such reallocation effect is more manifest with higher degree of heterogeneity. Note that as $\eta_i \to \infty$, i.e., as firm-level heterogeneity vanishes, the effect of aggregate output Y_t on cutoff productivity Z^*_{it} wanes. Equation (3.2), on the other hand, states that, given (K_t, L_t), a positive sector-specific TFP shock A_{it} and an improvement in allocation efficiency Z^*_{it} contribute to higher aggregate output. [9]

The joint work of production network and credit constraints create an aggregate financial multiplier in local dynamics. Combining equations (3.1) and (3.2) yields

$$\widehat{Y}_t = \mu \cdot \left[\sum_{i=1}^{N} \lambda_i \widehat{A}_{it} + \underline{\alpha}_K \widehat{K}_t + \underline{\alpha}_L \widehat{L}_t \right], \tag{3.3}$$

We illustrate in Figure 1 the mechanism through which credit constraints together with production network can generate an endogenous aggregate financial multiplier. An increase in output enables more productive firms to borrow more and produce more. As a result, it leads to reallocation that implies a higher endogenous sector-specific productivity $\widetilde{A}_{it}$ by raising the sector-specific productivity cutoff Z^*_i. This reallocation effect generates the sector-specific financial multiplier υ_i. On the other hand, the increase in sector-level productivity results in higher aggregate output through the production network, and this sector-level network effect is weighted by Domar weight λ_i. Thus, increased output Y leads to a ϱ-fold additional increase in output Y. The one-round financial multiplier ϱ is a Domar-weighted average of all sector-specific financial multiplier υ_i. This one-round financial multiplier will take effect in infinite rounds, as such an additional increase in output Y will lead to a new round of multiplication and so forth. Therefore, the total effect of the financial multiplier would be

[8]From equation (C.1) we know that in the steady state, the admissible parameter space is $\frac{\widetilde{\lambda}_i Y}{\Phi_i} > \eta_i$, i.e. sectoral fixed cost Φ_i is relatively unimportant compared with sectoral sales $\widetilde{\lambda}_i Y$ and/or within-sector heterogeneity is substantial.

[9]At the extensive margin, $\upsilon_i = 0$ if $\Phi_i = 0$, i.e., there is no fixed cost in sector i. The fixed cost is not essential for our results. Instead, it is obvious that what matters is the endogenously procyclical leverage ratio $\overline{B}_{it}/Y_t$, as indicated by equation (2.8). If $\Phi_i = 0$, but instead borrowing constraint Θ_{it} is endogenous, we can still have a positive effect of Y_t on Z^*_{it}, which is qualitatively similar to the findings in equation (3.1).

$\mu = \frac{1}{1-\varrho}$. For the convenience of exposition in the following sections, ϱ and μ are interchangeably referred to as economy-wide financial multipliers, as they are one-to-one mapped.

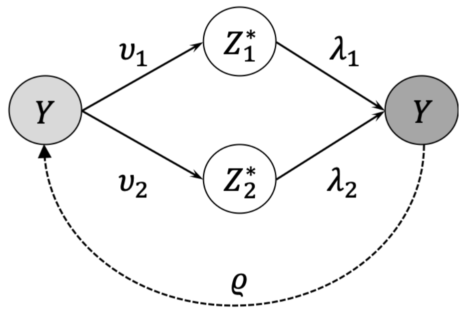


Figure 1: Aggregate Multiplier

Generalized Hulten Theorem. The effects of productivity shocks on output vary across sectors. There are two layers of amplification for uniform sector-level technology shocks: amplification through the production network and amplification through credit constraints.

Proposition 3.2. ***(Response of output to sector productivity shocks)*** *As shown by equations* (3.2) *and* (3.3),

$$\frac{\partial \ln Y_t}{\partial \ln A_{it}} = \lambda_i, \quad \sum_{i=1}^{N} \frac{\partial \ln Y_t}{\partial \ln A_{it}} = \sum_{i=1}^{N} \lambda_i \equiv \chi > 1 \quad \textit{(network multiplier)},$$
$$\frac{d \ln Y_t}{d \ln A_{it}} = \frac{\lambda_i}{1 - \sum_{i=1}^{N} v_i \lambda_i} = \mu\lambda_i > \lambda_i > \widetilde{\lambda}_i \qquad \textit{(network multiplier + financial multiplier)}.$$

Therefore the Hulten theorem fails to hold here since the first-order impact on output of a sectoral productivity shock is larger than that sector's sales share $\widetilde{\lambda}_i$. The Hulten theorem holds if and only if either of the three scenarios occurs: (a) $Var(Z_{it}) = 0$, i.e., $\eta_i \to \infty$; thus, there exists no firm heterogeneity. (b) $\Phi_i = 0$; thus, the leverage ratio $\overline{B}_{it}/Y_t$ is constant, and there is no misallocation due to financial frictions. (c) $\Theta_i \to \infty$; only the most productive firm produces. On the other hand, χ captures the percentage change in output in response to a uniform one-percent increase in all sectors' productivities. It captures a notion of returns to scale at the aggregate level. The amplification of this uniform productivity shock arises because goods are reproducible.

Aggregate Increasing Returns. It is direct to see that endogenous *increasing returns to scale (IRS)* emerges in reduced-form aggregate production function,

$$\frac{\partial \widehat{Y}_t}{\partial \widehat{K}_t} + \frac{\partial \widehat{Y}_t}{\partial \widehat{L}_t} = \mu > 1.$$

In particular, we do not need $\Phi_i > 0$ for all sectors to obtain IRS. Instead, the following holds,

$$\exists\ i \in \boldsymbol{N} \ \text{ such that } \ \Phi_i > 0 \quad \Longrightarrow \quad \mu = \left(1 - \sum_{i=1}^{N} \frac{\lambda_i \Phi_i}{\widetilde{\lambda}_i Y - \eta_i \Phi_i}\right)^{-1} > 1.$$

This condition describes that as long as the fixed cost is present in at least one sector, the endogenous IRS would always emerge in this economy. Next we will show that the equilibrium indeterminacy stems from endogenous IRS.

Equilibrium Indeterminacy. The reduced-form aggregate production in our model exhibits increasing returns to scale, which makes the economy prone to self-fulfilling sunspot-driven business cycles. Now we examine when sunspot-driven fluctuations are likely to occur. We focus on local dynamics around the steady state and abstract from fundamental shocks. The equilibrium can be summarized as the following log-linearized system of equations:

$$\begin{aligned}
\widehat{Y}_t &= \mu \left(\underline{\alpha}_K \widehat{K}_t + \underline{\alpha}_L \widehat{L}_t\right), \\
\widehat{L}_t &= \frac{1}{1+\gamma}\left(\widehat{Y}_t - \widehat{C}_{h,t}\right), \\
\dot{\widehat{K}}_t &= (\overline{\alpha}_K + \overline{\alpha}_L)\frac{Y}{K}\left(\widehat{Y}_t - \widehat{K}_t\right) - \frac{C_h}{K}\left(\widehat{C}_{h,t} - \widehat{K}_t\right), \\
\dot{\widehat{C}}_{h,t} &= \overline{\alpha}_K \frac{Y}{K}\left(\widehat{Y}_t - \widehat{K}_t\right).
\end{aligned}$$

Lemma 2. *Without fundamental shocks, the perfect foresight equilibrium can be equivalently summarized by the following log-linearized system of equations in $\widehat{Y}_t$ and $\widehat{K}_t$:*

$$\begin{bmatrix} \dot{\widehat{Y}}_t \\ \dot{\widehat{K}}_t \end{bmatrix} = \boldsymbol{J} \begin{bmatrix} \widehat{Y}_t \\ \widehat{K}_t \end{bmatrix}.$$

The Jacobian matrix is given by $\boldsymbol{J} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$. *Each element in the Jacobian matrix is*

$$\begin{aligned}
x_{11} &= \frac{Y}{K}\frac{\overline{\alpha}_K - \epsilon_{CK}(\overline{\alpha}_K + \overline{\alpha}_L)}{\epsilon_{CY}} + \frac{C_h}{K}\epsilon_{CK}, &\quad x_{12} &= \frac{C_h}{K}\frac{\epsilon_{CK}(\epsilon_{CK}-1)}{\epsilon_{CY}} + \frac{Y}{K}\frac{\epsilon_{CK}(\overline{\alpha}_K + \overline{\alpha}_L) - \overline{\alpha}_K}{\epsilon_{CY}}, \\
x_{21} &= \frac{Y}{K}(\overline{\alpha}_K + \overline{\alpha}_L) - \frac{C_h}{K}\epsilon_{CY}, &\quad x_{22} &= \frac{C_h}{K}(1 - \epsilon_{CK}) - \frac{Y}{K}(\overline{\alpha}_K + \overline{\alpha}_L),
\end{aligned}$$

where

$$\epsilon_{LK} = \frac{-\underline{\alpha}_K}{\underline{\alpha}_L}, \epsilon_{LY} = \frac{1}{\mu \underline{\alpha}_L}, \epsilon_{CK} = \frac{\underline{\alpha}_K(1+\gamma)}{\underline{\alpha}_L \mu}, \epsilon_{CY} = \frac{\underline{\alpha}_L \mu - 1 - \gamma}{\underline{\alpha}_L \mu}, \frac{Y}{K} = \frac{\delta + \rho_h}{\overline{\alpha}_K}, \frac{C_h}{K} = (\delta + \rho_h)\frac{\overline{\alpha}_K + \overline{\alpha}_L}{\overline{\alpha}_K} - \delta.$$

Following the seminal works by Benhabib and Farmer (1994) and the important extensions by

Wen (1998) and Benhabib and Wang (2013), among others, we know that the necessary and sufficient conditions for the existence of sunspot-driven fluctuations are $det(\boldsymbol{J}) > 0$ and $tr(\boldsymbol{J}) < 0$. These conditions lead us back to proposition 3.1 that regulates the parameters such that the aggregate financial multiplier falls in following range,

$$\max\left\{\mu_1^*, \mu_2^*\right\} < \mu < \mu_3^*.$$

Next, we will discuss some characteristics of the local dynamics. Since we are particularly interested in the role of the production network, we will sketch how the network weight and architecture alter the likelihood of sunspot-driven fluctuations. In the first exercise, we set the network structure to be symmetric and demonstrate how the intermediate input share can impact the formation of self-fulfilling business cycles. The second exercise turns to the asymmetric production network and inspects which network structure is more likely to drive the economy into multiple equilibria.

3.2 Altering the Network Weight

We now address how the intermediate input share alters the economy-wide financial multiplier. For computational convenience, we assume the following symmetric input-output table:

$$\boldsymbol{\alpha}_S = \boldsymbol{\alpha}_M \boldsymbol{\omega} = \begin{bmatrix} \alpha_M & 0 \\ 0 & \alpha_M \end{bmatrix} \begin{bmatrix} \omega & 1-\omega \\ 1-\omega & \omega \end{bmatrix}.$$

We fix the input-output linkage ω while altering the intermediate input share α_M. We assume that the ratio of total fixed costs to final goods output is constant to shut down the pro-cyclical leverage channel emphasized in Liu and Wang (2014) and only focus on the role played by the production network in driving indeterminacy. We further assume that the fixed cost Φ_i and financial constraint Θ_i are symmetric across sectors. Recall that the economy-wide financial multiplier is the weighted average of all sector-level financial multipliers,

$$\mu \equiv \left(1 - \sum_{i=1}^{N} \upsilon_i \lambda_i\right)^{-1} = (1-\varrho)^{-1},$$

where $\varrho = \sum_{i=1}^{N} \upsilon_i \lambda_i$, and the sector-level financial multiplier $\upsilon_i = \left(\widetilde{\lambda}_i / \phi_i - \eta_i\right)^{-1}$ hinges on the relative importance of fixed cost in this sector.

Figure 2 plots the response of the economy-wide financial multiplier to the change in the intermediate input share.[10] The key observation is that the impact of an increase in the intermediate input

[10]Parameters: $\phi_1 = \phi_2 = 0.065, \boldsymbol{\varphi} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}', \eta = 6, a = 0.2, \gamma = 0, \psi = 1, \delta = 0.025$. To calibrate Θ, match with Liu and Wang (2014), $\frac{\overline{B}}{Y-\Phi} = \frac{\overline{B}}{Y(1-\phi)} = 2.08$. Then $\Theta = \frac{2.08(1-\phi)\rho_e}{\xi(Z_1^*)+\xi(Z_2^*)-\phi}$.

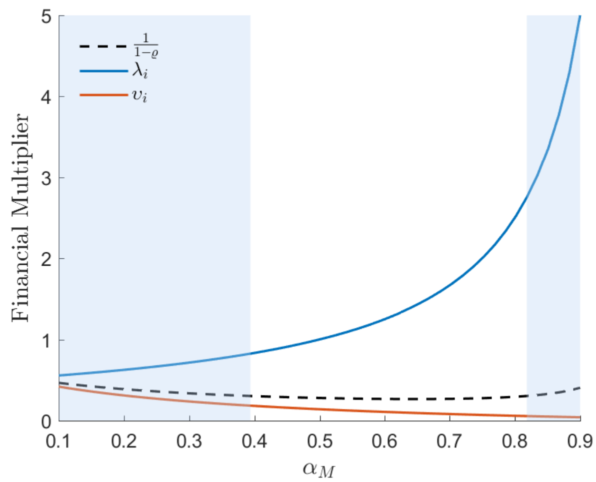


Figure 2: Financial multiplier when altering the intermediate input share

share α_M on the aggregate financial multiplier μ exhibits a U-shape. Since all sectors are symmetric, we can consider one sector as an example. The upward-sloping blue curve corresponds to the "size effect": an increase in the intermediate input share boosts the sales of intermediate firms, which is reflected by an increasing Domar weight. This tends to amplify the sector-level financial multiplier since intermediate firms that are subject to financial constraints expand. On the other hand, the downward-sloping red curve corresponds to the "diluting effect" because higher sales make the fixed cost play a less important role and thus dampen the procyclicality of the loan-to-output ratio. To be more precise, higher sales will dampen the sector-level financial multiplier $v_i = \left(\widetilde{\lambda}_i/\phi_i - \eta_i\right)^{-1}$.

These two forces are always pushing in opposite directions. However, the combination of the two determines the trend of economy-wide financial multiplier μ (represented by the black dashed curve). The curve is U-shaped, which indicates that the diluting effect dominates the size effect when the intermediate input share is not too large, and the relationship reverses when the intermediate input share reaches a sufficiently high level.

Furthermore, two blue shaded areas correspond to intermediate input shares that can lead the economy into multiple equilibria and are thus referred to as the "indeterminate region". Multiple equilibria can emerge either when the intermediate input share is too small or too large. When intermediate input share is too small, due to a relatively high fixed cost burden, intermediate firms are bound by too tight financial constraints. When intermediate input share is too large, intermediate firms that are subject to financial constraints are too large, and the size of aggregate financial multiplier is also large.

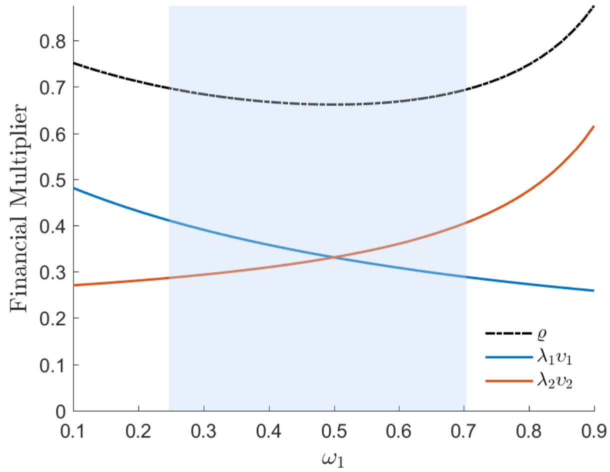


Figure 3: Financial multiplier when altering the network structure, $\omega_2 = 0.5$

3.3 Altering the Network Architecture

Another key target of this paper is to investigate the relevance of the production network structure to economic instability. Consider the effect when we modify the network structure but hold other aspects of the economy fixed. For computational convenience, we assume the following input-output table:

$$\boldsymbol{\alpha}_S = \boldsymbol{\alpha}_M \boldsymbol{\omega} = \begin{bmatrix} \alpha_M & 0 \\ 0 & \alpha_M \end{bmatrix} \begin{bmatrix} \omega_1 & 1 - \omega_1 \\ 1 - \omega_2 & \omega_2 \end{bmatrix}.$$

where the intermediate input share of a sector's own good is ω_i. We fix ω_2 and alter ω_1. A higher ω_1 means that sector 1 is a more "self-reliant" than "dependent" sector. A higher ω_1 also means that sectors are less interconnected. To isolate the effect of altering the network structure on the economy-wide financial multiplier, we again fix ϕ_i so that the relative fixed cost size does not affect the financial multiplier in this economy.

We illustrate how the aggregate financial multiplier changes when altering the network structure in Figure 3.[11] On the horizontal axis, we vary the value of ω_1. On the vertical axis, we report the economy-wide financial multiplier and all its decomposed effects. We address this question in three steps.

First, when ω_1 rises, sector 1 becomes more "self-reliant" and reduces its dependence on sector 2. As a result, the relative importance of sector 1 (reflected by Domar weight λ_1) increases, while the relative importance of sector 2 (reflected by Domar weight λ_2) is undermined. This is in line with the "size effect": the increase in ω_1 boosts the sales of sector 1 and reduces the sales of sector 2.

Second, as ω_1 increases, sector 1's financial multiplier v_1 decreases. This is due to the "diluting effect": the increase in sector 1's sales makes the fixed cost in sector 1 become relatively less important

[11] Parameters: $\phi_1 = \phi_2 = 0.095$, $\boldsymbol{\varphi} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}'$, $\eta = 6$, $\alpha_M = 0.5$, $b = 0.5$, $\gamma = 0$, $\psi = 1$, $\delta = 0.025$.

and effectively relaxes the financial constraint in sector 1. For sector 2, the reversal property holds, so the financial multiplier υ_2 is increasing in ω_1.

Third, the economy-wide financial multiplier ϱ is the Domar-weighted average of the sector-level financial multiplier. The blue curve corresponds to the joint effects of the "size effect" and "diluting effect" in sector 1. The blue curve is downward sloping, indicating that the "diluting effect" dominates the "size effect" in sector 1. The reverse is true for sector 2; see the red, upward-sloping curve. Thus, the combination of these two sectors' weighted average financial multiplier, or the black dashed curve, exhibits a U-shape. This helps us to see that when the network becomes more interconnected, it does not necessarily lead to a higher level of financial multiplier ϱ. The trend of the financial multiplier is determined by the relative forces of two sectors' weighted financial multipliers.

4. Critical Sector in Driving Sunspot Fluctuations

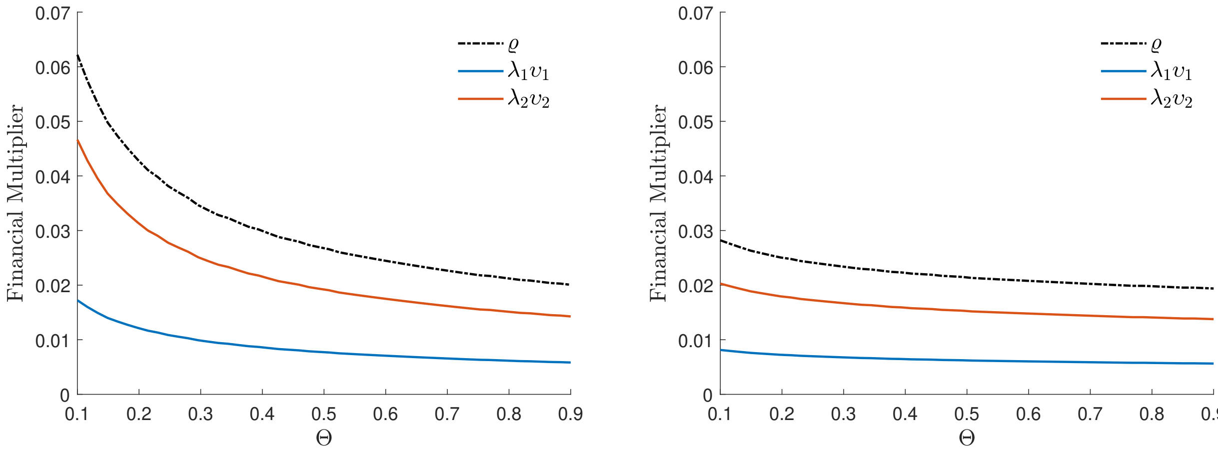
We now use a numerical example to illustrate that in order to avoid self-fulfilling fluctuations, how useful it is to relieve the financial constraints for sectors at different positions in the network. In particular, we tighten the financial constraints on different sectors and see which operation first triggers economy-wide multiple equilibria. The parameters used in this numerical exercise are to match with quarterly frequency[12]. To simplify our discussion, we will focus on a two-sector economy.

Heterogeneous Financial Shocks. We are curious about whether some sectors' financial constraints are more influential than others on the financial stability of the entire economy. This question is particularly relevant when the government has limited credit resources to allocate and has to prioritize which sector to subsidize. We consider an asymmetric two-sector economy to demonstrate the relative financial importance of different sectors in inducing self-fulfilling business cycles. For computational convenience, we assume the following input-output table:

$$\boldsymbol{\alpha}_S = \boldsymbol{\alpha}_M \boldsymbol{\omega} = \begin{bmatrix} \alpha_{M,1} & 0 \\ 0 & \alpha_{M,2} \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.9 & 0.1 \end{bmatrix}.$$

Sector 1 mainly uses its own product as the intermediate input, while sector 2 mainly uses the other sector's product as the intermediate input. Thus, in essence, sector 1 is in a more influential position in this production network. We refer to sector 1 as the upstream industry and sector 2 as the downstream industry. Now, given this network structure, we ask when financial shock hits one of the sector, which one is more likely to trigger sunspot fluctuations. We want to see the interaction between the asymmetric financial constraint and asymmetric production network. Other than the contract enforcement level Θ_i, both sectors are identical in fixed cost $\Phi_i = \Phi$. For the sector not being treated, we set $\Theta_i = 1$.

[12]Parameters: $\rho_h = 0.01$, $\rho_e = 0.02$, $\varkappa = 0.3$, $\alpha_{M,i} = 0.5$, $\delta = 0.025$, $\gamma = 0$, $\psi = 1$, $\eta_i = 6$, $\varphi = [0.5 \quad 0.5]$.



(a) Shock Θ_1 (b) Shock Θ_2

Figure 4: Aggregate financial multiplier when altering contract enforcement

We conduct two experiments by shocking sector 1 and sector 2's credit market efficiency values Θ_1 and Θ_2, respectively. Figure 4 shows the results of these financial shock treatments. On the horizontal axis, we vary the financial constraint of a chosen sector. On the vertical axis, we report the economy-wide financial multiplier and its decomposition.

We begin with Figure 4a, which corresponds to the response of the economy-wide financial multiplier when tightening the financial constraint in the upstream sector 1. As the level of Θ_1 moves from 0.9 to 0.1, the working capital constraint is exogenously tightened. Consequently, the Domar-weighted financial multiplier of the upstream sector increases; see the blue downward-sloping curve. However, this is not the whole story: the downstream sector 2 is also affected through the input-output linkages. In fact, the Domar-weighted financial multiplier of the downstream sector ($\lambda_2 \upsilon_2$) is not only shifted but also shifted faster than that in sector 1; see the red, steeper downward-sloping curve in the same graph. In this sense, treatment in the upstream sector can easily "alter" the financial landscape in the downstream sector. The joint forces of the two sectors shape the economy-wide financial multiplier represented by the black dashed line, which is quite steep, suggesting a very sensitive response of the economy-wide financial multiplier to the upstream sector's financial constraint.

Instead, if we tighten the financial constraint in the downstream sector 2, which is shown in Figure 4b, the Domar-weighted financial multipliers $\lambda_1 \upsilon_1$ and $\lambda_2 \upsilon_2$ in the two sectors co-move – they both increase in response to a negative sector 2 financial shock. However, since the downstream sector has a low Domar weight in this economy, an exogenous tightening of its financial constraint leads to a minor change in its own sector; see the red, relatively flat line. Moreover, since sector 2 occupies a relatively inferior node in the production network, it is less powerful in "altering" the financial landscape of sector 1; see the blue, even flatter line. Consequently, the combination of two sectors' effects leads to a highly insensitive response by the economy-wide financial multiplier, reflected by a much flatter black dashed line. We conclude that the upstream sector has a larger effect on the economy-wide financial

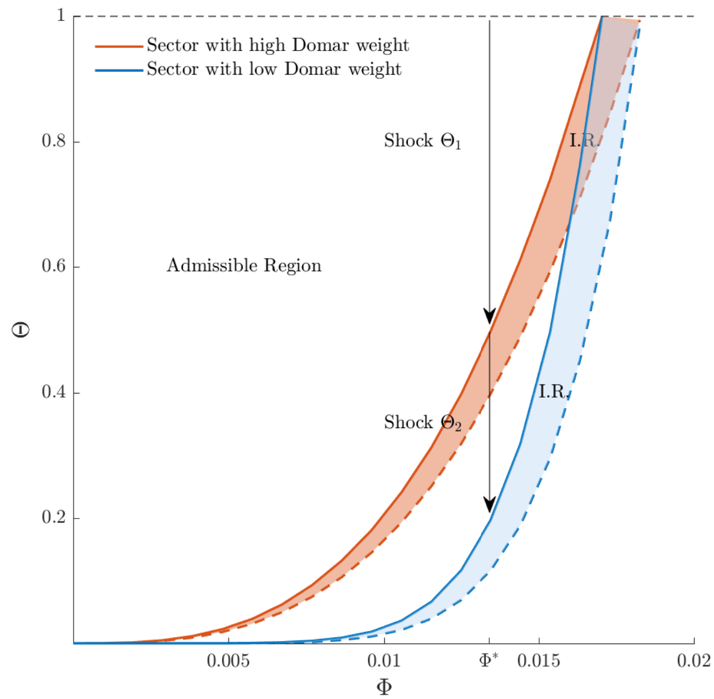


Figure 5: Indeterminacy: financial shock to sectors with different levels of influence

multiplier. This also translates into a larger impact in inducing economy-wide instability.

Let us visualize how self-fulfilling fluctuations are connected with sector-level financial constraints through Figure 5. The horizontal axis corresponds to different levels of fixed costs, and the vertical axis corresponds to different levels of financial constraints in the treatment sector, holding the other sector's financial constraint fixed at 1; a lower Θ_i means a tighter financial constraint. The red shaded area is the set of (Θ_1, Φ) pairs that can lead the economy to multiple equilibria if the financial shock affects sector 1. On the other hand, the blue shaded area collects (Θ_2, Φ) pairs that can drive the economy to multiple equilibria if the financial shock takes effects on sector 2. We call these shaded areas "indeterminate regions"(I.R.). The areas to the left of these two indeterminate regions, including the indeterminate regions, are "admissible regions".

Given one particular level of fixed cost Φ^*, if we tighten the financial constraint, i.e., decrease Θ from 1 towards 0, then tightening the upstream sector's financial constraint more easily drives the economy into the "indeterminate region" than tightening the downstream sector's financial constraint; the line with an arrow first reaches red shaded indeterminate region when Θ_1 declines to approximately 0.5 and then reaches the blue shaded "indeterminate region" when Θ_2 declines to approximately 0.2. This reveals the importance of financial abundance for the upstream sector in stabilizing the economy. We conclude that if the government has limited resources and attention, it should focus on improving financial market efficiency for upstream sectors.

5. Quantifying likelihood of multiple equilibria

We now apply the theory to evaluate the likelihood of self-fulfilling business cycles in the US at annual frequency. We discuss the main data source we rely on, and then describe the procedure we go through to obtain the aggregate financial multiplier. Lastly, we show the patterns of financial multiplier in the US economy which is a direct indicator for the likelihood of self-fulfilling business cycles.

5.1 Data

After harmonizing among various data sources, the data covers the period of time from 1998 to 2020. We use input-output account data by Bureau of Economic Analysis (BEA) for 15 sectors. These data contains information on the uses of commodities by intermediate and final users. We use input-output data to estimate the intermediate commodity input share $\alpha_{S,ijt}$. In addition, we use integrated industry-level production account (KLEMS) data by BEA for 63 industries. We aggregate 63 industries into 15 sectors according to NAICS classification as shown in details in appendix D. These data provides information on industry level compensation on capital and labor, and information on industry level gross output. We use KLEMS data to estimate the labor share $\alpha_{L,it}$ and capital share $\alpha_{K,it}$. Furthermore, we estimate the financial constraint for each sector using data by Compustat. From Compustat data, we can collect information on firm level current debt, long term debt and asset value. As a result, we can construct firm level debt ratio which is the total debt over asset value. The industry level debt ratio is computed through aggregating the firm level debt ratios weighted by firm sales. For simplicity, we assume that firm will always borrow to the limit, thus the industry level debt ratio is an approximation for the industry specific financial constraint θ_{it}. The details of the quantitative exercise will be delegated to appendix D.

Other calibrated parameters are summarized in table 1. ρ_h is set at 0.04 to match the annual household discount rate of 0.96. ρ_e is set at 0.08 to match the annual entrepreneur discount rate of 0.92. The annual capital depreciation rate is set at 0.1 and the Frisch elasticity is set to match the micro evidence at 0.5.

5.2 Sunspot fluctuations

Figure 6 plots the time series of aggregate financial multiplier μ using red dots. Corresponding to proposition 3.1, the blue shaded area represents the region of indeterminacy defined by $\mu \in \left[\underline{\mu}, \overline{\mu}\right]$ where $\underline{\mu} = \max\left\{\mu_1^*, \mu_2^*\right\}$ and $\overline{\mu} = \mu_3^*$. As shown in the figure, our computed upper bound of the indeterminacy region is too high to be relevant. On the other hand, the lower bound is given by the blue line and expands indeterminacy region in the period of pre-2000, $2003-2007$ and $2011-2014$. Meanwhile, the aggregate financial multiplier is at high level in the period of pre- 2000 and $2007-2015$.

Table 1: Parameters used for quantitative exercise

Parameter	Description	Value
ρ_h	household discount rate	0.04
ρ_e	entrepreneur discount rate	0.08
δ	capital depreciation rate	0.1
γ	inverse of Frisch elasticity	2
ψ	effort disutility	1

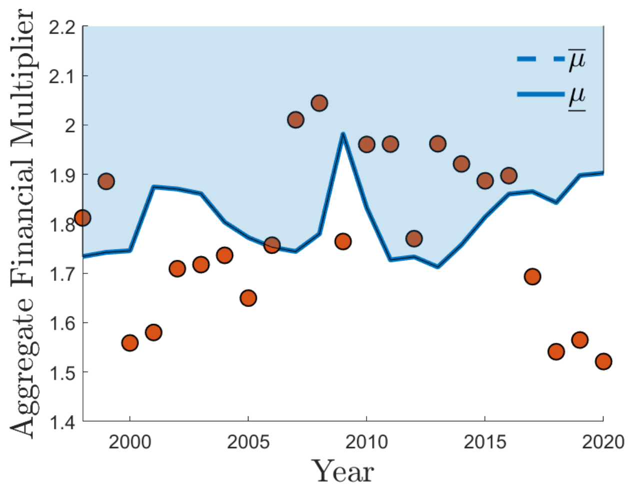
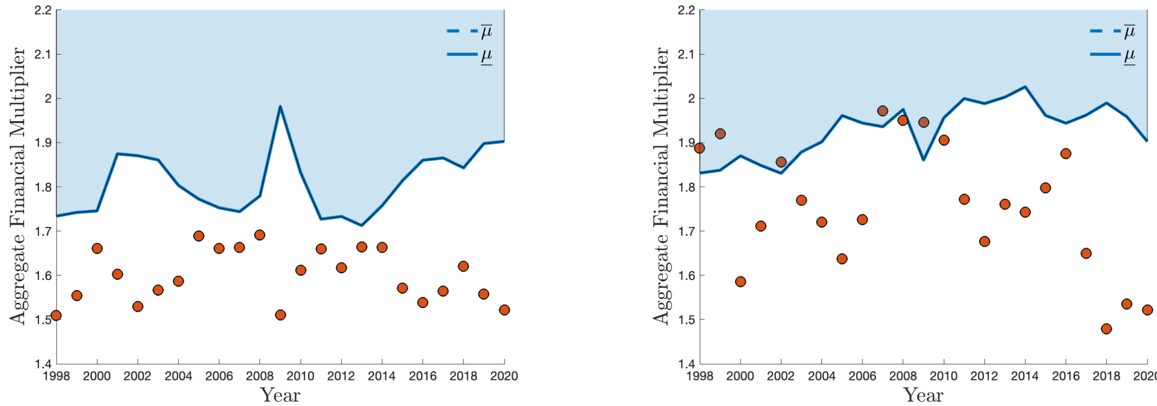


Figure 6: Likelihood of self-fulfilling business cycles

Combining these two pieces of information, leading to higher likelihood of self-fulfilling business cycles in period of pre-2000 (the Internet bubble boom) and around 2007 – 2015 (the Global Financial Crisis).

Figure 7 decomposes the "size effect" and "diluting effect" in shaping the self-fulfilling business cycles. Recall that the aggregate financial multiplier is determined by $\boldsymbol{\upsilon}$ and $\boldsymbol{\lambda}$. Panel 7a sets the sectoral financial multipliers $\boldsymbol{\upsilon}$ at year 2020's level and the variation in μ is driven by the change in Domar weights $\boldsymbol{\lambda}$. The red dots representing the aggregate financial multipliers do not fall into the indeterminacy region in panel 7a, suggesting that "size effect" is not the main factor that leads to self-fulfilling business cycles. On the other hand, panel 7b sets the Domar weights $\boldsymbol{\lambda}$ at year 2020's level: the variation in μ is all driven by the change in sectoral financial multiplier. The red dots fall into the indeterminacy region in years roughly in accordance with that in figure 6, indicating that "diluting effect" is the key driving force for the self-fulfilling business cycles.

The anatomy of the quantitative results shows that the sectoral financial multipliers play a more



(a) Set sectoral financial multipliers $\boldsymbol{\upsilon}$ at 2020 level

(b) Set Domar weights $\boldsymbol{\lambda}$ at 2020 level

Figure 7: Decompose "size effect" and "diluting effect"

important role in exposing the economy to endogenous fluctuations. This explains why the aggregate financial multiplier is higher and remains high after year 2007. Following the Global Financial Crisis, the credit condition exacerbates for firms, resulting in high sectoral financial multipliers in the following years.

6. Conclusion

Are certain production network weights and structures more likely to drive the economy into a self-fulling equilibrium? Are certain sectors' levels of financial market efficiency more important for the economy's stability? This paper makes a contribution in answering these questions.

In this paper, we develop a flexible and tractable framework that studies the interactions of financial frictions and input-output linkages and their joint impact on sunspot-driven fluctuations. In the benchmark model, we incorporate the production network into a self-fulfilling business cycle model. We show that the economy permits a simple but rich aggregation, which allows an analytical analysis of the financial multiplier and network multiplier.

By introducing the production network, we investigate the endogenous aggregate financial multiplier and find that it is linked with the weight and structure of the production network. The impact of the intermediate input share on the economy-wide financial multiplier is U-shaped. In addition, multiple self-fulfilling equilibria can arise when we alter the network structure. We also show that when tightening the credit constraint for a particular sector, its effect in terms of inducing self-fulfilling business cycles hinges on the relative importance of this sector: a sector with higher Domar weight is more likely to trigger sunspot-driven fluctuations after a financial depression. Then in the quantitative exercise we evaluate the likelihood of self-fulfilling business cycles over the 22 years and find the sectoral financial multiplier is the main reason that leads economy into sunspot fluctuations.

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Appendices

A. Characterizing the Economy

A. 1 Timing

Time is continuous. However, to provide a better illustration, we examine the results "within a given moment". Entrepreneurs first have to decide for each firm whether to stay in business or exit. If they stay, a fixed operating cost has to be paid. Only after the cost is paid are entrepreneurs able to observe each firm's individual draw of productivity. Seeing the productivity shock, the entrepreneurs then decide whether a firm will produce or not based on the expected profit. If a firm operates, working capital is loaned to the firm. Once production is completed, the entrepreneur chooses to repay or default on the working capital loan.

A. 2 Household Euler Equation

The current-value Hamiltonian for households is given by

$$\mathcal{H}\left(K_h, C_h, L, \lambda\right) = \log\left(C_h\right) - \psi \frac{L^{1+\gamma}}{1+\gamma} + \lambda\left[-\delta K_h + RK_h + WL - C_h\right].$$

To generate the sufficient conditions for the optimum, following equations have to be satisfied,

$$\begin{aligned}
\frac{\mathcal{H}\left(K_h, C_h, L, \lambda\right)}{C} &= \frac{1}{C_h} - \lambda = 0, \\
\frac{\mathcal{H}\left(K_h, C_h, L, \lambda\right)}{L} &= -\psi L^{\gamma} + \lambda W = 0, \\
\dot{\lambda} = \rho_h \lambda - \frac{\mathcal{H}\left(K_h, C_h, L, \lambda\right)}{K} &= \rho_h \lambda - \lambda\left[R - \delta\right] = 0.
\end{aligned}$$

These imply that household Euler equations are given by

$$\frac{\dot{C}_h}{C_h} = R - \delta - \rho_h, \tag{A.1}$$

$$\frac{W}{C_h} = \psi L^{\gamma}. \tag{A.2}$$

A. 3 Intermediate Firm Value and Working Capital Constraint

Profit. Define the profit of a firm ι in sector i as $\Pi_{it}(\iota)$, which depends on the entrepreneur's decision on the firm being operative or inactive after observing the idiosyncratic shock $Z_{it}(\iota)$,

$$\Pi_{it}(\iota) = \max_{\{O,\mathcal{I}\}} \{\Pi_{O,it}(\iota), \Pi_{\mathcal{I},it}(\iota)\},$$

$\{O, \mathcal{I}\}$ denotes the choice between operating and staying idle. $\Pi_{O,it}(\iota)$ is the momentary profit of an operating firm,

$$\Pi_{O,it}(\iota) = \max_{K_{it}(\iota), L_{it}(\iota), S_{ijt}(\iota)} P_{it} O_{it}(\iota) - R_t K_{it}(\iota) - W_t L_{it}(\iota) - \sum_{j=1}^{N} P_{jt} S_{ijt}(\iota) - \Phi_i. \tag{A.3}$$

On the other hand, if the firm stays inactive for some period (because of a low productivity draw) but remains in the business in the hope of becoming profitable later, the momentary "profit" (actually loss from overhead costs) is $\Pi_{\mathcal{I},it}(\iota)$,

$$\Pi_{\mathcal{I},it}(\iota) = -\Phi_i.$$

Continuation Value. Define V_{it} as the continuation value for the firm to stay in business, which is also the ex ante value of a firm before paying the fixed cost. Let $V_{it}(\iota)$ be an individual firm's value; then, $V_{it} = \int V_{it}(\iota)d\iota = \int V(Z_{it}(\iota))dF(Z_{it}(\iota))$. The continuation value takes the following recursive form:

$$V_{it} = -\Phi_i + \max\{V_{\mathcal{I},it}(\iota), V_{\mathcal{O},it}(\iota)\}.$$

$V_{\mathcal{I},it}(\iota)$ is the value of an inactive firm, and $V_{\mathcal{O},it}(\iota)$ is the value of an operative firm. As there is no current output, the value of an inactive firm $V_{\mathcal{I},it}(\iota)$ only comes from the future discounted value,

$$V_{\mathcal{I},it}(\iota) = \max\left\{e^{-\rho\, dt}\mathbb{E}\left[\frac{\Lambda_{e,it+dt}}{\Lambda_{e,it}}V_{it+dt}\right], 0\right\}.$$

The stochastic discount factor is given by $\frac{\Lambda_{e,it+dt}}{\Lambda_{e,it}}$, where $\Lambda_{e,it} = u'(C_{e,it})$ is the marginal utility of the entrepreneur owning sector i. The value of an operative firm $V_{\mathcal{O},it}(\iota)$ comes from both current profit and future value,

$$V_{\mathcal{O},it}(\iota) = \max\left\{V_{\mathcal{D},it}(\iota), V_{\mathcal{C},it}(\iota)\right\}.$$

$\{\mathcal{D}, \mathcal{C}\}$ denotes the choice between defaulting and committing. The value of a firm committing to its working capital loan is $V_{\mathcal{C},it}(\iota)$ and that of a defaulting firm is $V_{\mathcal{D},it}(\iota)$.

Intermediate Firm Working Capital Constraint. Since firms have limited liability, the credit offered is tailored to provide a sufficient repayment incentive. If the firm chooses to commit to the loan, its value is given by

$$V_{\mathcal{C},it}(\iota) = \left(P_{it}O_{it}(\iota) - R_tK_{it}(\iota) - W_tL_{it}(\iota) - \sum_{j=1}^{N} P_{jt}S_{ijt}(\iota)\right)dt + \max\left\{e^{-\rho_e dt}\frac{\Lambda_{e,it+dt}}{\Lambda_{e,it}}V_{it+dt}, 0\right\}.$$

If the firm chooses to default, then it can seize all the revenue at the risk of being caught. The event of being caught is a Poisson process that arrives with intensity Θ_i. Should a defaulting firm be caught, it is deprived of all future access to credit; thus, the value of the defaulting firm is given by

$$V_{\mathcal{D},it}(\iota) = P_{it}O_{it}(\iota)dt + (1 - \Theta_i dt)\max\left\{e^{-\rho_e dt}\frac{\Lambda_{e,it+dt}}{\Lambda_{e,it}}V_{it+dt}, 0\right\}.$$

To guarantee there is no default on the equilibrium path, the incentive compatibility (IC) condition requires that $V_{\mathcal{C},it}(\iota) \geq V_{\mathcal{D},it}(\iota)$, i.e.,

$$\Theta_i e^{-\rho_e dt}\max\left\{\frac{\Lambda_{e,it+dt}}{\Lambda_{e,it}}V_{it+dt}, 0\right\}dt \geq \left(R_tk_{it} + W_tl_{it} + \sum_{j=1}^{N} P_{jt}s_{ijt}\right)dt.$$

In the limit, the IC condition takes a very simple form:

$$R_t K_{it}(\iota) + W_t L_{it}(\iota) + \sum_{j=1}^{N} P_{jt} S_{ijt}(\iota) \leq \Theta_i V_{it} \equiv \overline{B}_{it}. \tag{A.4}$$

A. 4 Intermediate Firm Policy

We will show in Lemma 3 that with the presence of the working capital constraints, there exists a cutoff productivity Z^*_{it} for each sector, above which the firm participates in production; otherwise, the firm stays inactive.

Lemma 3. ***(Individual Firm Policy Functions)*** *For each sector i, there exists a cutoff value Z^*_{it}, such that firms' optimal capital input, labor input and intermediate goods input are given by*

$$K_{it}(\iota) = \frac{\alpha_{K,i}\overline{B}_{it}}{R}\mathbb{1}_{\{Z_{it}(\iota)\geq Z^*_{it}\}}, \quad L_{it}(\iota) = \frac{\alpha_{L,i}\overline{B}_{it}}{W}\mathbb{1}_{\{Z_{it}(\iota)\geq Z^*_{it}\}}, \quad S_{ijt}(\iota) = \frac{\alpha_{S,ij}\overline{B}_{it}}{P_j}\mathbb{1}_{\{Z_{it}(\iota)\geq Z^*_{it}\}},$$

and firms' optimal borrowing amount and output are as follows:

$$B_{it}(\iota) = \overline{B}_{it}\mathbb{1}_{\{Z_{it}(\iota)\geq Z^*_{it}\}}, \quad O_{it}(\iota) = \frac{Z_{it}(\iota)}{Z^*_{it}}\frac{\overline{B}_{it}}{P_{it}}\mathbb{1}_{\{Z_{it}(\iota)\geq Z^*_{it}\}}.$$

The cutoff productivity Z^*_{it} is determined by

$$Z^*_{it} \equiv \frac{1}{A_{it}}\left(\frac{R_t/P_{it}}{\alpha_{K,i}}\right)^{\alpha_{K,i}}\left(\frac{W_t/P_{it}}{\alpha_{L,i}}\right)^{\alpha_{L,i}}\prod_{j=1}^{N}\left(\frac{P_{jt}/P_{it}}{\alpha_{S,ij}}\right)^{\alpha_{S,ij}}.$$

Proof. Denote $\Pi_{it}(\iota)$ as the profit of firm ι in sector i

$$\Pi_{it}(\iota) = \max_{K_{it}(\iota),L_{it}(\iota),S_{ijt}(\iota)} P_{it}O_{it}(\iota) - R_t K_{it}(\iota) - W_t L_{it}(\iota) - \sum_{j=1}^{N} P_{jt}S_{ijt}(\iota).$$

Given the capital rental rate R_t, the wage rate W_t, and the vector of intermediate goods prices $\{P_{it}\}$, if the optimal choice is an interior solution, then the first order conditions lead to

$$\begin{aligned} K_{it}(\iota) &= \frac{\eta_i\alpha_i P_{it}O_{it}(\iota)}{R_t}, \\ L_{it}(\iota) &= \frac{\eta_i(1-\alpha_i)P_{it}O_{it}(\iota)}{W_t}, \\ S_{ijt}(\iota) &= \frac{(1-\eta_i)\varpi_{ij}P_{it}O_{it}(\iota)}{P_{jt}}. \end{aligned}$$

The necessary condition for interior solutions is

$$O_{it}(\iota) = A_{it}Z_{it}(\iota)\left[\left(\frac{\eta_i\alpha_i P_{it}O_{it}(\iota)}{R_t}\right)^{\alpha_i}\left(\frac{\eta_i(1-\alpha_i)P_{it}O_{it}(\iota)}{W_t}\right)^{1-\alpha_i}\right]^{\eta_i}\left[\prod_{j=1}^{N}\left(\frac{(1-\eta_i)\varpi_{ij}P_{it}O_{it}(\iota)}{P_{jt}}\right)^{\varpi_{ij}}\right]^{1-\eta_i}.$$

This necessary condition for interior solution implies that there exists an unique cutoff value for firm-specific productivity shock Z^*_{it},

$$Z^*_{it} = \frac{1}{A_{it}} \left(\frac{R_t/P_{it}}{\eta_i \alpha_i} \right)^{\eta_i \alpha_i} \left(\frac{W_t/P_{it}}{\eta_i (1-\alpha_i)} \right)^{\eta_i (1-\alpha_i)} \prod_{j=1}^{N} \left(\frac{P_{jt}/P_{it}}{(1-\eta_i)\varpi_{ij}} \right)^{(1-\eta_i)\varpi_{ij}} .$$

If a firm has unlimited access to the credit market, then it will produce infinite amount if its draw of firm specific productivity $Z_{it}(\iota) > Z^*_{it}$, while stay out of production if $Z_{it}(\iota) < Z^*_{it}$. And Any amount of production can be supported if $Z_{it}(\iota) = Z^*_{it}$.

The idea behind is very simple. The marginal revenue of producing one unit of intermediate goods is P_{it}, while the corresponding marginal cost is denoted by $\Psi_{it}(\iota)$

$$\Psi_{it}(\iota) = \frac{1}{A_{it} Z_{it}(\iota)} \left(\frac{R_t}{\alpha_{K,i}} \right)^{\alpha_{k,i}} \left(\frac{W_t}{\alpha_{L,i}} \right)^{\alpha_{L,i}} \prod_{j=1}^{N} \left(\frac{P_{jt}}{\alpha_{S,ij}} \right)^{\alpha_{S,ij}} ,$$

where $\alpha_{K,i} = \eta_i \alpha_i$, $\alpha_{L,i} = \eta_i(1-\alpha_i)$, $\alpha_{S,ij} = (1-\eta_i)\varpi_{ij}$. The individual firm will only participate in production if their marginal cost of producing does not exceed their marginal revenue.

With the presence of credit constraints, the gross production expense is capped by the borrowing limit $\overline{B}_{it}$. The borrowing limit effectively controls the scale of production for firm whose productivity exceeds the threshold Z^*_i. Given a borrowing limit $\overline{B}_i$, the total expenses are fixed. Then the individual firm output is given by

$$O_{it}(\iota) = A_{it} Z_{it}(\iota) \left[\left(\frac{\alpha_{K,i} \overline{B}_{it}}{R_t} \right)^{\alpha_{K,i}} \left(\frac{\alpha_{L,i} \overline{B}_{it}}{W_t} \right) \alpha_{L,i} \prod_{j=1}^{N} \left(\frac{\alpha_{S,ij} \overline{B}_{it}}{P_{jt}} \right)^{\alpha_{S,ij}} \right] = \frac{Z_{it}(\iota)}{Z^*_{it}} \frac{\overline{B}_{it}}{P_{it}} .$$

∎

A. 5 Expenditure Share: Sector-level and Aggregate-level

The marginal cost of intermediate goods input i must be equal to its marginal benefit in final goods production,

$$X_{it} = \frac{\varphi_i Y_t}{P_{it}} . \tag{A.5}$$

Moreover, from (A.5), sector i's sales to final goods producers are a fraction of its total sales, $X_{it} = \frac{\varphi_i}{\lambda_{it}} O_{it}$.

Sector Aggregation. We can examine the allocation of resources across firms. Define the sector-level aggregate inputs as K_{it}, L_{it}, S_{ijt} and sector-level output as O_{it}. By the law of large numbers, the sector-level variables are given by

$$K_{it} = (1 - F_i(Z^*_{it})) \frac{\alpha_{L,i} \overline{B}_{it}}{R} , \quad L_{it} = (1 - F_i(Z^*_{it})) \frac{\alpha_{L,i} \overline{B}_{it}}{W} , \quad S_{ijt} = (1 - F_i(Z^*_{it})) \frac{\alpha_{S,ij} \overline{B}_{it}}{P_j} ,$$

$$O_{it} = \int \frac{Z_{it}(\iota)}{Z^*_{it}} \frac{\overline{B}_{it}}{P_{it}} d\iota = \frac{\overline{B}_{it}}{P_{it} Z^*_{it}} \int_{Z^*_{it}} Z_{it}(\iota) dF(Z_{it}(\iota)). \tag{A.6}$$

Given the sector input $\{K_{it}, L_{it}, S_{ijt}\}$, the sector output can also be written as

$$O_{it} = \int_0^1 A_{it} Z_{it}(\iota) \left[K_{it}(\iota)^{\alpha_{K,i}} L_{it}(\iota)^{\alpha_{L,i}} \prod_{j=1}^{N} S_{ijt}^{\alpha_{S,ij}} \right] d\iota = \widetilde{A}_{it} \cdot K_{it}^{\alpha_{K,i}} L_{it}^{\alpha_{L,i}} \prod_{j=1}^{N} S_{ijt}^{\alpha_{S,ij}}, \tag{A.7}$$

where we denote $\widetilde{A}_{it} = A_{it}\mathbb{E}_i\left(Z_{it}|Z_{it} \geq Z_{it}^*\right) > A_{it}$ as the endogenous sector-specific productivity.

Sector Expenditure Shares. Sector i's expenditures on inputs from sector j are $P_{jt}S_{ijt}$. Sector i's expenditure from sector j is a fraction of sector i's revenue, and accordingly, one can see that sector i's expenditures on intermediate goods, labor and capital are given by

$$P_{jt}S_{ijt} = \widetilde{\alpha}_{S,ijt}P_{it}O_{it}, \quad W_t L_{it} = \widetilde{\alpha}_{L,it}P_{it}O_{it}, \quad R_t K_{it} = \widetilde{\alpha}_{K,it}P_{it}O_{it}, \tag{A.8}$$

where $\widetilde{\alpha}_{S,ijt} \equiv \alpha_{S,ij}\frac{Z_{it}^*}{\mathbb{E}_i\left(Z_{it}(\iota)|Z_{it}(\iota)\geq Z_{it}^*\right)}$, $\widetilde{\alpha}_{L,it} \equiv \alpha_{L,i}\frac{Z_{it}^*}{\mathbb{E}_i\left(Z_{it}(\iota)|Z_{it}(\iota)\geq Z_{it}^*\right)}$, $\widetilde{\alpha}_{K,it} \equiv \alpha_{K,i}\frac{Z_{it}^*}{\mathbb{E}_i\left(Z_{it}(\iota)|Z_{it}(\iota)\geq Z_{it}^*\right)}$ are defined as endogenous expenditure shares.

Final Expenditure Shares. The economy-wide aggregate demands for labor and capital are given by

$$L_t = \sum_{i=1}^{N} L_{it} = \sum_{i=1}^{N} \widetilde{\alpha}_{L,it}\frac{P_{it}O_{it}}{Y_t}\frac{Y_t}{W_t} = \overline{\alpha}_{L,t}\frac{Y_t}{W_t}, \tag{A.9}$$

$$K_t = \sum_{i=1}^{N} K_{it} = \sum_{i=1}^{N} \widetilde{\alpha}_{K,it}\frac{P_{it}O_{it}}{Y_t}\frac{Y_t}{R_t} = \overline{\alpha}_{K,t}\frac{Y_t}{R_t}. \tag{A.10}$$

Let $\overline{\alpha}_{L,t}, \overline{\alpha}_{K,t}$ be the total expenditure shares of labor and capital,

$$\overline{\alpha}_{L,t} \equiv \widetilde{\boldsymbol{\lambda}}_t' \widetilde{\boldsymbol{\alpha}}_{L,t}, \ \overline{\alpha}_{K,t} \equiv \widetilde{\boldsymbol{\lambda}}_t' \widetilde{\boldsymbol{\alpha}}_{K,t},$$

A. 6 The Law of Motion for Aggregate Capital

Note that, the aggregate law of motion of K_t is given by

$$\dot{K}_t = -\delta K_t + I_t,$$

where $K_t = K_{h,t} + \sum_{i=1}^{N} K_{e,it}$, and $I_t = I_{h,t} + \sum_{i=1}^{N} I_{e,it}$ and I_t can be obtained from the aggregate resource constraint

$$I_t = Y_t - \sum_{i=1}^{N} \Phi_i - C_{h,t} - C_{e,t},$$

and thus aggregate entrepreneur's consumption is given by aggregate budget constraint, and by final expenditure share (A.9) and (A.10),

$$
\begin{aligned}
C_{e,t} &= \sum_{i=1}^{N} C_{e,it} \\
&= \sum_{i=1}^{N} \left[P_{it} O_{it} - R_t K_{it} - W_t L_{it} - \sum_{j=1}^{N} P_{jt} S_{ijt} - \Phi_i \right] \\
&= \sum_{i=1}^{N} \left[P_{it} \left(X_{it} + \sum_{j=1}^{N} S_{jit} \right) - R_t K_{it} - W_t L_{it} - \sum_{j=1}^{N} P_{jt} S_{ijt} - \Phi_i \right] \\
&= \sum_{i=1}^{N} P_{it} X_{it} - W_t L_t - R_t K_t - \sum_{i=1}^{N} \Phi_i \\
&= \sum_{i=1}^{N} P_{it} \frac{\varphi_i Y_t}{P_{it}} - W_t L_t - R_t K_t - \sum_{i=1}^{N} \Phi_i \\
&= Y_t - W_t L_t - R_t K_t - \sum_{i=1}^{N} \Phi_i \\
&= \left(1 - \widetilde{\lambda}_t' \widetilde{\alpha}_{K,t} - \widetilde{\lambda}_t' \widetilde{\alpha}_{L,t} \right) Y_t - \sum_{i=1}^{N} \Phi_i \\
&= (1 - \overline{\alpha}_{K,t} - \overline{\alpha}_{L,t}) Y_t - \sum_{i=1}^{N} \Phi_i .
\end{aligned}
$$

Consequently, the mapping between $\dot{K}_t$ and K_t can be obtained as

$$
\dot{K}_t = -\delta K_t + (\overline{\alpha}_{K,t} + \overline{\alpha}_{L,t}) Y_t - C_{h,t}.
$$

B. Proofs

B. 1 Proof of Proposition 2.1

Proof. Recall from collateral constraint (A.4),

$$
\overline{B}_{it} = \Theta_i V_{it}.
$$

Note that from Section 2, $V_{it} = \int V_{it}(\iota) d\iota = \int V(Z_{it}(\iota)) dF(Z_{it}(\iota))$ denotes the ex ante value of the firm. Obviously V_{it} is also the continuation value of sector i which is the present value of the firm. From (2.5) and (2.6), $C_{e,it} = D_{e,it} = \int_0^1 \Pi_{it}(\iota) d\iota$, V_{it} is given by

$$
V_{it} = \int_0^1 \left(\int_t^\infty e^{-\rho_e (s-t)} \frac{u'(C_{e,is})}{u'(C_{e,it})} \Pi_{e,is}(\iota) ds \right) d\iota = \int_t^\infty e^{-\rho_e (s-t)} \frac{C_{e,it}}{C_{e,is}} \left(\int_0^1 \Pi_{e,is}(\iota) d\iota \right) ds = \frac{C_{e,it}}{\rho_e}.
$$

In the equilibrium, $C_{e,it}$ is the aggregate dividends payment from all firms. And by sectoral demand derived earlier (A.8) as well as the definition of Domar Weight, the consumption of sector i entrepreneur is related to the final output in the following way,

$$C_{e,it} = \int_0^1 \Pi_{it}(\iota)d\iota = \int_0^1 \left(P_{it}O_{it}(\iota) - R_t K_{it}(\iota) - W_t L_{it}(\iota) - \sum_{j=1}^{N} P_{jt} S_{ijt}(\iota) - \Phi_i \right) d\iota$$

$$= P_{it}O_{it} - R_t K_{it} - W_t L_{it} - \sum_{j=1}^{N} P_{jt} S_{ijt} - \Phi_i$$

$$= P_{it}O_{it} \left(1 - \widetilde{\alpha}_{K,it} - \widetilde{\alpha}_{L,it} - \sum_{j=1}^{N} \widetilde{\alpha}_{S,ijt} \right) - \Phi_i$$

$$= \xi(Z^*_{it})Y_t - \Phi_i.$$

Note that since $\widetilde{\alpha}_{K,it}, \widetilde{\alpha}_{L,it}, \widetilde{\alpha}_{S,ijt}, \widetilde{\lambda}_{it}$ are all functions of Z^*_t, so is ξ.c

$$\xi(Z^*_{it}) = \widetilde{\lambda}_{it} \left(1 - \widetilde{\alpha}_{K,it} - \widetilde{\alpha}_{L,it} - \sum_{j=1}^{N} \widetilde{\alpha}_{S,ijt} \right).$$

Therefore, the sectoral debt limit is related to the final output as such,

$$\frac{\overline{B}_{it}}{Y_t} = \frac{\Theta}{\rho_{e\ i}} \left[\xi\left(Z^*_{it}\right) - \frac{\Phi_i}{Y_t} \right] \equiv f(Z^*_{it})$$

We will show that with Y_t given, $\xi\left(Z^*_{it}\right)$ is a weakly decreasing function in Z^*_{it} and so is $f(Z^*_{it})$. For now, denote $\nu_i = \mathbb{E}_i \left(\frac{Z_{it}}{\widetilde{Z}^*_i} | Z_{it} \geq \widetilde{Z}^*_{it} \right)$.

$$\xi_i = \sum_{k=0}^{\infty} \left(\widetilde{\boldsymbol{\alpha}}'_{S,t}\right)^k \boldsymbol{\varphi} \left(1 - \frac{1}{\nu_i} \right)$$

On the other hand, if we look from credit demand side, (A.6) shows that loan-to-GDP ratio is related to cutoff productivity Z^*_t,

$$\frac{\overline{B}_{it}}{Y_t} = \frac{\widetilde{\lambda}_{it} Z^*_{it}}{\int_{Z^*_{it}} Z_{it}(\iota) dF(Z_{it}(\iota))} \equiv g(Z^*_{it}).$$

Notice that $g(Z^*_{it})$ is an increasing function is Z^*_i. As $\widetilde{\lambda}_{it}$ is weakly increasing in Z^*_{it}. Whereas $\frac{Z^*_{it}}{\int_{Z^*_{it}} Z_{it}(\iota)dF(Z_{it}(\iota))} = \frac{1}{\mathbb{E}_i\left(\frac{Z_{it}}{\widetilde{Z}^*_i}|Z_{it} \geq \widetilde{Z}^*_{it}\right)\left(1-F(Z^*_{it})\right)}$, where $\mathbb{E}_i \left(\frac{Z_{it}}{\widetilde{Z}^*_i} | Z_{it} \geq \widetilde{Z}^*_{it} \right)$ is weakly decreasing in Z^*_{it}, and $\left(1 - F(Z^*_{it})\right)$ is decreasing in Z^*_{it} too.

∎

B. 2 Proof of Proposition 2.2

Proof. Recall from (A.7) that the sector output is,

$$O_{it} = A_{it}\mathbb{E}_i\left(Z_{it}(\iota)|Z_{it}(\iota) \geq Z^*_{it}\right) \cdot K_{it}^{\alpha_{K,i}} L_{it}^{\alpha_{L,i}} \prod_{j=1}^{N} S_{ijt}^{\alpha_{S,ij}}, \tag{B.1}$$

and from (A.8) that demand for intermediate goods j is,

$$S_{ijt} = \widetilde{\alpha}_{S,ijt}\frac{P_{it}O_{it}}{P_{jt}} = \widetilde{\alpha}_{S,ijt}\frac{P_{it}O_{it}/Y_t}{P_{jt}O_{jt}/Y_t}O_{jt} = \widetilde{\alpha}_{S,ijt}\frac{\widetilde{\lambda}_{it}}{\widetilde{\lambda}_{jt}}O_{jt},$$

If we take log on both sides of equation (B.1),

$$\begin{aligned}
\ln O_{it} &= \ln \mathbb{E}_i\left(Z_{it}(\iota)|Z_{it}(\iota) \geq Z^*_{it}\right) + \ln A_{it} \\
&+ \alpha_{K,i}\ln K_{it} + \alpha_{L,i}\ln L_{it} + \sum_{j=1}^{N}\alpha_{S,ij}\ln S_{ijt} \\
&= \ln \mathbb{E}_i\left(Z_{it}(\iota)|Z_{it}(\iota) \geq Z^*_{it}\right) + \ln A_{it} \\
&+ \alpha_{K,i}\ln k_{it} + \alpha_{L,i}\ln l_{it} + \sum_{j=1}^{N}\alpha_{S,ij}\ln\left(\widetilde{\alpha}_{S,ijt}\frac{\widetilde{\lambda}_{it}}{\widetilde{\lambda}_{jt}}\right) \\
&+ \alpha_{K,i}\ln K_t + \alpha_{L,i}\ln L_t + \sum_{j=1}^{N}\alpha_{S,ij}\ln O_{jt},
\end{aligned}$$

Then we know that

$$\ln \boldsymbol{O}_t = \widetilde{\boldsymbol{a}}_t + \boldsymbol{\alpha}_K \ln K_t + \boldsymbol{\alpha}_L \ln L_t + \boldsymbol{\alpha}_S \ln \boldsymbol{O}_t + \boldsymbol{c}^o_t,$$

where an entry in $\widetilde{\boldsymbol{a}}_t$ is

$$\widetilde{a}_{it} \equiv \ln \mathbb{E}_i\left(Z_{it}|Z_{it} \geq Z^*_{it}\right) + \ln A_{it}, \tag{B.2}$$

and an entry in $\boldsymbol{c}^o_t$ is

$$c^o_{it} \equiv \alpha_{K,i}\ln k_{it} + \alpha_{L,i}\ln l_{it} + \sum_{j=1}^{N}\alpha_{S,ij}\ln\left(\widetilde{\alpha}_{S,ijt}\frac{\widetilde{\lambda}_{it}}{\widetilde{\lambda}_{jt}}\right).$$

In turn, the sector output can be expressed as follows,

$$\begin{aligned}
\ln \boldsymbol{O}_t &= (\boldsymbol{I} - \boldsymbol{\alpha_S})^{-1}\left(\widetilde{\boldsymbol{a}}_t + \boldsymbol{\alpha}_K \ln K_t + \boldsymbol{\alpha}_L \ln L_t + \boldsymbol{c}^o_t\right) \\
&= \left[\left(\boldsymbol{I} - \boldsymbol{\alpha}'_{\boldsymbol{S}}\right)^{-1}\right]'\left(\widetilde{\boldsymbol{a}}_t + \boldsymbol{\alpha}_K \ln K_t + \boldsymbol{\alpha}_L \ln L_t + \boldsymbol{c}^o_t\right) \\
&= \boldsymbol{\Psi}'_t\left(\widetilde{\boldsymbol{a}}_t + \boldsymbol{\alpha}_K \ln K_t + \boldsymbol{\alpha}_L \ln L_t + \boldsymbol{c}^o_t\right).
\end{aligned}$$

Since $\widetilde{\lambda}_{it} = \frac{P_{it}O_{it}}{Y_t} = \frac{\varphi_j O_{it}}{X_{it}}$, we know that

$$\ln X_{it} = \ln O_{it} + \ln\left(\frac{\varphi_i}{\widetilde{\lambda}_{it}}\right),$$

more compactly,

$$\ln \boldsymbol{X}_t = \ln \boldsymbol{O}_t + \boldsymbol{c}_t^x,$$

where $c_{it}^x = \ln\left(\frac{\varphi_i}{\widetilde{\lambda}_{it}}\right)$. Also by final goods production function $Y_t = \prod_{i=1}^{N} X_{it}^{\varphi_i}$,

$$\begin{aligned}
\ln Y_t &= \sum_{j=1}^{N} \varphi_j \ln X_{jt} \\
&= \boldsymbol{\varphi}' \ln \boldsymbol{X}_t \\
&= \boldsymbol{\varphi}' \ln \boldsymbol{O}_t + \boldsymbol{\varphi}' \boldsymbol{c}_t^x \\
&= \boldsymbol{\varphi}' \boldsymbol{\Psi}_t' \left(\widetilde{\boldsymbol{a}}_t + \boldsymbol{\alpha}_K \ln K_t + \boldsymbol{\alpha}_L \ln L_t + \boldsymbol{c}_t^o\right) + \boldsymbol{\varphi}' \boldsymbol{c}_t^x \\
&= \boldsymbol{\lambda}_t' \left(\widetilde{\boldsymbol{a}}_t + \boldsymbol{\alpha}_K \ln K_t + \boldsymbol{\alpha}_L \ln L_t + \boldsymbol{c}_t^o\right) + \boldsymbol{\varphi}' \boldsymbol{c}_t^x,
\end{aligned}$$

and by (B.2),

$$\boldsymbol{\lambda}_t' \widetilde{\boldsymbol{a}}_t = \sum_{i=1}^{N} \lambda_i \widetilde{a}_{it} = \sum_{i=1}^{N} \ln \left(A_{it} \mathbb{E}_i \left(Z_{it} | Z_{it} \geq Z_{it}^*\right)\right)^{\lambda_i} = \ln \left[\prod_{i=1}^{N} \left(A_{it} \mathbb{E}_i \left(Z_{it} | Z_{it} \geq Z_{it}^*\right)\right)^{\lambda_i}\right].$$

Combine these together, we get aggregate output,

$$Y_t = A_t K_t^{\boldsymbol{\lambda}' \boldsymbol{\alpha}_K} L_t^{\boldsymbol{\lambda}' \boldsymbol{\alpha}_L},$$

where

$$\begin{aligned}
A_t &= \exp\left[\boldsymbol{\lambda}_t' \widetilde{\boldsymbol{a}}_t \left(\boldsymbol{Z}_t^*\right) + \boldsymbol{\lambda}' \boldsymbol{c}_t^o \left(\boldsymbol{Z}_t^*\right) + \boldsymbol{\varphi}' \boldsymbol{c}_t^x \left(\boldsymbol{Z}_t^*\right)\right] \\
&= \prod_{i=1}^{N} \left[A_{it} \mathbb{E}_i \left(Z_{it}(\iota) | Z_{it}(\iota) \geq Z_{it}^*\right) k_{it}^{\alpha_{K,i}} l_{it}^{\alpha_{L,i}} \prod_{j=1}^{N} \left(\widetilde{\alpha}_{S,ijt} \frac{\widetilde{\lambda}_{it}}{\widetilde{\lambda}_{jt}}\right)^{\alpha_{S,ij}}\right]^{\lambda_i} \cdot \prod_{i=1}^{N} \left(\frac{\varphi_i}{\widetilde{\lambda}_{it}}\right)^{\varphi_i}.
\end{aligned}$$

Notice that the factor intensities add up to 1,

$$\begin{aligned}
\boldsymbol{\lambda}' \boldsymbol{\alpha}_K + \boldsymbol{\lambda}' \boldsymbol{\alpha}_L &= \boldsymbol{\lambda}' \left(\boldsymbol{\alpha}_K + \boldsymbol{\alpha}_L\right) \\
&= \boldsymbol{\varphi}' \left(\boldsymbol{I} - \boldsymbol{\alpha}_S\right)^{-1} \left(\boldsymbol{\alpha}_K + \boldsymbol{\alpha}_L\right) \\
&= \boldsymbol{\varphi}' \sum_{n=0}^{\infty} \boldsymbol{\alpha}_S^n \left(\boldsymbol{1} - \boldsymbol{\alpha}_S \boldsymbol{1}\right) \\
&= \boldsymbol{\varphi}' \boldsymbol{1} + \boldsymbol{\varphi}' \sum_{n=1}^{\infty} \boldsymbol{\alpha}_S^n \boldsymbol{1} - \boldsymbol{\varphi}' \sum_{n=0}^{\infty} \boldsymbol{\alpha}_S^n \boldsymbol{\alpha}_S \boldsymbol{1} \\
&= \boldsymbol{\varphi}' \boldsymbol{1} = 1.
\end{aligned}$$

Besides, note that since the idiosyncratic productivities conform to the Pareto distribution, $\widetilde{\alpha}_{S,ijt}, \widetilde{\alpha}_{L,it}, \widetilde{\alpha}_{K,it}$,

$\widetilde{\lambda}_{it}$, λ_{it}, ξ_{it}, k_{it}, l_{it}, $\underline{\alpha}_K$, $\underline{\alpha}_L$ are all constants such that

$$\widetilde{\alpha}_{S,ij} = \alpha_{S,ij}\underline{Z}_i, \quad \widetilde{\alpha}_{L,i} = \alpha_{L,i}\underline{Z}_i, \quad \widetilde{\alpha}_{K,i} = \alpha_{K,i}\underline{Z}_i,$$
$$k_i = \frac{K_{it}}{K_t} = \frac{\widetilde{\alpha}_{K,i}\widetilde{\lambda}_i}{\sum_{i=1}^N \widetilde{\alpha}_{K,i}\widetilde{\lambda}_i}, \quad l_i = \frac{L_{it}}{L_t} = \frac{\widetilde{\alpha}_{L,i}\widetilde{\lambda}_i}{\sum_{i=1}^N \widetilde{\alpha}_{L,i}\widetilde{\lambda}_i}.$$

Similarly, the firms' expected profit not excluding fixed costs is also a constant,

$$\xi_i = \left[1 - \left(\alpha_{K,i} + \alpha_{L,i} + \sum_{j=1}^N \alpha_{S,ij}\right)\underline{Z}_i\right]\widetilde{\lambda}_i = \left(1 - \underline{Z}_i\right)\widetilde{\lambda}_i = \widetilde{\lambda}_i/\eta_i.$$

∎

B. 3 Proof of Lemma 1

Proof. In this appendix we will show how we log linearize the $Z_i^* - Y$ system.

Denote $\widehat{x}_t = \log X_t - \log X$, so that $X_t = X(1 + \widehat{x}_t)$. Also, when $X \to 0$, we have $\exp X = 1 + X$.

$$Z_{it}^* = \left[\frac{\Theta_{it}}{\rho_e(\eta_i - 1)}\left(1 - \frac{\eta_i \Phi_i}{\widetilde{\lambda}_i Y_t}\right)\right]^{1/\eta_i}\underline{Z}_i,$$

Linearizing above equation around the steady state, denote $X_t = \frac{\Theta_{it}}{\rho_e(\eta_i-1)} - \frac{\Theta_{it}}{Y_t}\frac{\eta_i\Phi_i}{\widetilde{\lambda}_i\rho_e(\eta_i-1)}$,

$$\begin{aligned}
Z_i^*(1 + \widehat{Z}_{it}^*) &= X^{1/\eta_i}\left(1 + \widehat{X}_t\right)^{\frac{1}{\eta_i}}\underline{Z}_i \\
&= X^{1/\eta_i}\left(\exp \widehat{X}_t\right)^{\frac{1}{\eta_i}}\underline{Z}_i \\
&= X^{1/\eta_i}\exp\left(\frac{1}{\eta_i}\widehat{X}_t\right)\underline{Z}_i \\
&= X^{1/\eta_i}\left(1 + \frac{1}{\eta_i}\widehat{X}_t\right)\underline{Z}_i.
\end{aligned}$$

Since in the steady state, the equation degenerate into

$$Z_i^* = X^{1/\eta_i}\underline{Z}_i.$$

Thus we have

$$\widehat{Z}_{it}^* = \frac{1}{\eta_i}\widehat{X}_t.$$

On the other hand,

$$
\begin{aligned}
X(1+\widehat{X}_t) &= \frac{\Theta}{\rho_e(\eta_i-1)}\left(1+\widehat{\Theta}_{it}\right) - \frac{\Theta}{Y}\frac{\eta_i\Phi_i}{\widetilde{\lambda}_i\rho_e(\eta_i-1)}\frac{1+\widehat{\Theta}_{it}}{1+\widehat{Y}_t} \\
&= \frac{\Theta}{\rho_e(\eta_i-1)}\left(1+\widehat{\Theta}_{it}\right) - \frac{\Theta}{Y}\frac{\eta_i\Phi_i}{\widetilde{\lambda}_i\rho_e(\eta_i-1)}\frac{\exp\widehat{\Theta}_{it}}{\exp\widehat{Y}_t} \\
&= \frac{\Theta}{\rho_e(\eta_i-1)}\left(1+\widehat{\Theta}_{it}\right) - \frac{\Theta}{Y}\frac{\eta_i\Phi_i}{\widetilde{\lambda}_i\rho_e(\eta_i-1)}\exp\left(\widehat{\Theta}_{it}-\widehat{Y}_t\right) \\
&= \frac{\Theta}{\rho_e(\eta_i-1)}\left(1+\widehat{\Theta}_{it}\right) - \frac{\Theta}{Y}\frac{\eta_i\Phi_i}{\widetilde{\lambda}_i\rho_e(\eta_i-1)}\left(1+\widehat{\Theta}_{it}-\widehat{Y}_t\right).
\end{aligned}
$$

Again in the steady state,

$$
X = \frac{\Theta_i}{\rho_e(\eta_i-1)} - \frac{\Theta_i}{Y}\frac{\eta_i\Phi_i}{\widetilde{\lambda}_i\rho_e(\eta_i-1)}.
$$

Then we can get linearized equation,

$$
X\widehat{X}_t = \frac{\Theta_i}{\rho_e(\eta_i-1)}\widehat{\Theta}_{it} - \frac{\Theta_i}{Y}\frac{\eta_i\Phi_i}{\widetilde{\lambda}_i\rho_e(\eta_i-1)}\left(\widehat{\Theta}_{it}-\widehat{Y}_t\right).
$$

$$
\widehat{X}_t = \widehat{\Theta}_{it} + \frac{\Theta_i}{XY}\frac{\eta_i\Phi_i}{\widetilde{\lambda}_i\rho_e(\eta_i-1)}
$$

In turn we have,

$$
\begin{aligned}
\widehat{Z}^*_{it} &= \frac{1}{\eta_i}\left[\widehat{\Theta}_{it} + \frac{\Theta_i}{XY}\frac{\eta_i\Phi_i}{\widetilde{\lambda}_i\rho_e(\eta_i-1)}\widehat{Y}_t\right] \\
&= \frac{1}{\eta_i}\left[\widehat{\Theta}_{it} + \frac{\frac{\Theta_i}{Y}\frac{\eta_i\Phi_i}{\widetilde{\lambda}_i\rho_e(\eta_i-1)}}{\frac{\Theta_i}{\rho_e(\eta_i-1)} - \frac{\Theta_i}{Y}\frac{\eta_i\Phi_i}{\widetilde{\lambda}_i\rho_e(\eta_i-1)}}\widehat{Y}_t\right] \\
&= \frac{1}{\eta_i}\left[\widehat{\Theta}_{it} + \frac{\eta_i\Phi_i}{Y\widetilde{\lambda}_i-\eta_i\Phi_i}\widehat{Y}_t\right] \\
&= \frac{1}{\eta_i}\widehat{\Theta}_{it} + \upsilon_i\widehat{Y}_t,
\end{aligned}
$$

where

$$
\upsilon_i = \frac{\Phi_i}{Y\widetilde{\lambda}_i-\eta_i\Phi_i} = \frac{\phi_i}{\widetilde{\lambda}_i-\eta_i\phi_i} = \left(\frac{\widetilde{\lambda}_iY}{\Phi_i}-\eta_i\right)^{-1}.
$$

Since we assume that the credit tightness is fixed at Θ_i, the log-linearized cutoff productivity degenerates into

$$
\widehat{Z}^*_{it} = \upsilon_i\widehat{Y}_t.
$$

Now we log-linearize the aggregate production function

$$
Y_t = \zeta\prod_{i=1}^{N}\left[A_{it}\frac{Z^*_{it}}{\underline{Z}_i}\right]^{\lambda_i}K_t^{\alpha_K}L_t^{\alpha_L}.
$$

Take log on both sides,

$$\log Y_t = \log \zeta + \sum_{i=1}^{N} \lambda_i \left[\log A_{it} + \log Z_{it}^* - \log \underline{Z}_i\right] + \underline{\alpha}_K \log K_t + \underline{\alpha}_L \log L_t.$$

At the steady state,

$$\log Y = \log \zeta + \sum_{i=1}^{N} \lambda_i \left[\log A_i + \log Z_i^* - \log \underline{Z}_i\right] + \underline{\alpha}_K \log K + \underline{\alpha}_L \log L.$$

Find the difference of the two equations,

$$\widehat{Y}_t = \sum_{i=1}^{N} \lambda_i \widehat{A}_{it} + \sum_{i=1}^{N} \lambda_i \widehat{Z}_{it}^* + \underline{\alpha}_K \widehat{K}_t + \underline{\alpha}_L \widehat{L}_t.$$

∎

B. 4 Proof of Lemma 2

Proof. The evolvement of $C_{h,t}$ and K_t are respectively,

$$\begin{aligned} \frac{\dot{C}_{h,t}}{C_{h,t}} &= \overline{\alpha}_K \frac{Y_t}{K_t} - \delta - \rho_h, \\ \frac{\dot{K}_t}{K_t} &= -\delta + (\overline{\alpha}_K + \overline{\alpha}_L) \frac{Y_t}{K_t} - \frac{C_{h,t}}{K_t}. \end{aligned}$$

The local dynamics around steady state can be summarized by the following system of equations,

$$\begin{aligned} \widehat{Y}_t &= \mu \left(\underline{\alpha}_K \widehat{K}_t + \underline{\alpha}_L \widehat{L}_t\right), \\ \widehat{L}_t &= \frac{1}{1+\gamma} \left(\widehat{Y}_t - \widehat{C}_{h,t}\right), \\ \dot{\widehat{K}}_t &= (\overline{\alpha}_K + \overline{\alpha}_L) \frac{Y}{K} \left(\widehat{Y}_t - \widehat{K}_t\right) - \frac{C_h}{K} \left(\widehat{C}_{h,t} - \widehat{K}_t\right), \\ \dot{\widehat{C}}_{h,t} &= \overline{\alpha}_K \frac{Y}{K} \left(\widehat{Y}_t - \widehat{K}_t\right). \end{aligned}$$

This system can be reduced into law of motion of $(\widehat{Y}_t, \widehat{K}_t)$ pair,

$$\begin{aligned} \widehat{C}_{h,t} &= \frac{\underline{\alpha}_K (1+\gamma)}{\underline{\alpha}_L} \widehat{K}_t - \frac{1+\gamma-\underline{\alpha}_L \mu}{\underline{\alpha}_L \mu} \widehat{Y}_t = \epsilon_{CK} \widehat{K}_t + \epsilon_{CY} \widehat{Y}_t \\ \widehat{L}_t &= \frac{-\underline{\alpha}_K}{\underline{\alpha}_L} \widehat{K}_t + \frac{1}{\mu \underline{\alpha}_L} \widehat{Y}_t = \epsilon_{LK} \widehat{K}_t + \epsilon_{LY} \widehat{Y}_t \\ \dot{\widehat{K}}_t &= \left[(\overline{\alpha}_K + \overline{\alpha}_L) \frac{Y}{K} - \frac{C_h}{K} \epsilon_{CY}\right] \widehat{Y}_t - \left[(\overline{\alpha}_K + \overline{\alpha}_L) \frac{Y}{K} + \frac{C_h}{K} (\epsilon_{CK} - 1)\right] \widehat{K}_t, \\ \dot{\widehat{C}}_{h,t} &= \epsilon_{CK} \dot{\widehat{K}}_t + \epsilon_{CY} \dot{\widehat{Y}}_t \\ \dot{\widehat{Y}}_t &= \left[\frac{Y}{K} \frac{\overline{\alpha}_K - \epsilon_{CK}(\overline{\alpha}_K + \overline{\alpha}_L)}{\epsilon_{CY}} + \frac{C_h}{K} \epsilon_{CK}\right] \widehat{Y}_t + \left[\frac{C_h}{K} \frac{\epsilon_{CK} (\epsilon_{CK} - 1)}{\epsilon_{CY}} + \frac{Y}{K} \frac{\epsilon_{CK} (\overline{\alpha}_K + \overline{\alpha}_L) - \overline{\alpha}_K}{\epsilon_{CY}}\right] \widehat{K}_t \end{aligned}$$

where

$$\epsilon_{LK}=\frac{-\underline{\alpha}_K}{\underline{\alpha}_L},\ \ \epsilon_{LY}=\frac{1}{\mu\underline{\alpha}_L},\ \ \epsilon_{CK}=\frac{\underline{\alpha}_K(1+\gamma)}{\underline{\alpha}_L},\ \ \epsilon_{CY}=-\frac{1+\gamma-\underline{\alpha}_L\mu}{\underline{\alpha}_L\mu},$$
$$Y/K=\frac{\delta+\rho_h}{\overline{\alpha}_K},\qquad C_h/K=(\bar{\alpha}_K+\bar{\alpha}_L)\frac{Y}{K}-\delta=\left(\frac{\overline{\alpha}_K+\overline{\alpha}_L}{\overline{\alpha}_K}\right)(\delta+\rho_h)-\delta.$$

We can vectorize the system,

$$\begin{bmatrix}\dot{\widehat{Y}}_t\\ \dot{\widehat{K}}_t\end{bmatrix}=\boldsymbol{J}\begin{bmatrix}\widehat{Y}_t\\ \widehat{K}_t\end{bmatrix}.$$

Each element in the Jacobian matrix $\boldsymbol{J}=\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix}$ is given by

$$x_{11}=\frac{Y}{K}\frac{\overline{\alpha}_K-\epsilon_{CK}(\overline{\alpha}_K+\overline{\alpha}_L)}{\epsilon_{CY}}+\frac{C_h}{K}\epsilon_{CK},\qquad x_{12}=\frac{C_h}{K}\frac{\epsilon_{CK}\left(\epsilon_{CK}-1\right)}{\epsilon_{CY}}+\frac{Y}{K}\frac{\epsilon_{CK}\left(\overline{\alpha}_K+\overline{\alpha}_L\right)-\overline{\alpha}_K}{\epsilon_{CY}},$$
$$x_{21}=\frac{Y}{K}\left(\overline{\alpha}_K+\overline{\alpha}_L\right)-\frac{C_h}{K}\epsilon_{CY},\qquad x_{22}=\frac{C_h}{K}\left(1-\epsilon_{CK}\right)-\frac{Y}{K}\left(\overline{\alpha}_K+\overline{\alpha}_L\right).$$

∎

B. 5 Proof of Proposition 3.1

Proof. The necessary and sufficient condition for indeterminate equilibrium to exist is given by

$$det(\boldsymbol{J})>0,\ \ tr(\boldsymbol{J})<0$$

Notice that

$$det(\boldsymbol{J})=x_{11}x_{22}-x_{12}x_{21}=\frac{(1+\gamma)(\delta+\rho_h)\left[\delta-(\delta+\rho_h)\frac{\overline{\alpha}_K+\overline{\alpha}_L}{\overline{\alpha}_K}\right]\left(1-\mu\underline{\alpha}_K\right)}{1+\gamma-\mu\underline{\alpha}_L},$$

$$tr(\boldsymbol{J})=x_{11}+x_{22}=\frac{(1+\gamma)(\delta+\rho_h)\frac{\overline{\alpha}_K+\overline{\alpha}_L}{\overline{\alpha}_K}\mu\underline{\alpha}_K-\delta(1+\gamma)-\rho_h\mu\underline{\alpha}_L}{1+\gamma-\mu\underline{\alpha}_L}.$$

Obviously,

$$(\delta+\rho_h)\frac{\overline{\alpha}_K+\overline{\alpha}_L}{\overline{\alpha}_K}>\delta+\rho_h>\delta,$$

thus $det(\boldsymbol{J})>0$ is equivalent to

$$1-\mu\underline{\alpha}_K>0,\ \ 1+\gamma-\mu\underline{\alpha}_L<0,$$

and $tr(\boldsymbol{J})<0$ is equivalent to

$$\left[(1+\gamma)(\delta+\rho_h)\frac{\overline{\alpha}_K+\overline{\alpha}_L}{\overline{\alpha}_K}\underline{\alpha}_K-\rho_h\underline{\alpha}_L\right]\mu>\delta(1+\gamma).$$

Therefore, we can obtain from the above three inequalities the range of μ that is both necessary and sufficient for the indeterminate equilibrium to exist.

$$\max\left\{\mu_1^*, \mu_2^*\right\} < \mu < \mu_3^*$$

where

$$\mu_1^* = \frac{1+\gamma}{\underline{\alpha}_L}$$
$$\mu_2^* = \frac{\delta(1+\gamma)}{\frac{\overline{\alpha}_K+\overline{\alpha}_L}{\overline{\alpha}_K}(\delta+\rho_h)(1+\gamma)\underline{\alpha}_K - \rho_h\underline{\alpha}_L}$$
$$\mu_3^* = \frac{1}{\underline{\alpha}_K}$$

∎

C. Steady State and Global Dynamics

C. 1 Steady State Output

From Proposition 2.2, the equilibrium aggregate production function is given by

$$Y_t = \zeta \prod_{i=1}^{N}\left[A_{it}^{\lambda_i}\left[\frac{\Theta_i}{\rho_e(\eta_i-1)}\left(1-\frac{\eta_i\Phi_i}{\widetilde{\lambda}_i Y_t}\right)\right]^{\lambda_i/\eta_i}\right]K_t^{\underline{\alpha}_K}L_t^{\underline{\alpha}_L}.$$

Suppose all the sectors share the same level of productivity heterogeneity, i.e. $\eta_i = \eta$ for all i. In the steady state, the aggregate output solves the following nonlinear equation

$$Y^{\eta+\sum\lambda_i-\eta\underline{\alpha}_K}\left(\frac{Y}{K}\right)^{\eta\underline{\alpha}_K} = \left(\zeta\prod_{i=1}^{N}A_i^{\lambda_i}L^{\underline{\alpha}_L}\right)^{\eta}\cdot\prod_{i=1}^{N}\left[\frac{\Theta_i}{\rho_e(\eta-1)}\left(Y-\frac{\eta\Phi_i}{\widetilde{\lambda}_i}\right)\right]^{\lambda_i},$$

where

$$\zeta = \prod_{i=1}^{N}\left[k_i^{\alpha_{K,i}}l_i^{\alpha_{L,i}}\prod_{j=1}^{N}\left(\widetilde{\alpha}_{S,ij}\frac{\widetilde{\lambda}_i}{\widetilde{\lambda}_j}\right)^{\alpha_{S,ij}}\right]^{\lambda_i}\cdot\prod_{i=1}^{N}\left(\frac{\varphi_i}{\widetilde{\lambda}_i}\right)^{\varphi_i}, \quad \underline{\alpha}_K = \boldsymbol{\lambda}'\boldsymbol{\alpha}_K, \quad \underline{\alpha}_L = \boldsymbol{\lambda}'\boldsymbol{\alpha}_L.$$

The Euler equation of the households (A.1) together with the aggregate demand for capital (A.10) yields the steady-state capital rental rate, which is inversely related to the output-capital ratio,

$$R = \overline{\alpha}_K\frac{Y}{K} = \delta+\rho_h, \quad \frac{Y}{K} = \frac{\delta+\rho_h}{\overline{\alpha}_K}.$$

Moreover, from the law of motion for aggregate capital (2.11), we can obtain the aggregate investment rate, which yields the steady-state consumption-capital ratio,

$$I_t = (\overline{\alpha}_K+\overline{\alpha}_L)Y_t - C_{h,t} = \delta K_t, \quad \frac{C_h}{K} = (\overline{\alpha}_k+\overline{\alpha}_l)\frac{Y}{K}-\delta.$$

The aggregate labor demand (A.9) together with the labor supply Euler equation (A.2) yields the steady-state wage and steady-state labor,

$$W=\overline{\alpha}_L\frac{Y}{L}=\psi L^{\gamma}C_h,\quad L=\left(\frac{\overline{\alpha}_L}{\psi}\frac{Y/K}{C_h/K}\right)^{\frac{1}{1+\gamma}}=\left(\frac{\overline{\alpha}_L}{\psi}\frac{1}{(\overline{\alpha}_K+\overline{\alpha}_L)-\frac{\delta\overline{\alpha}_K}{\delta+\rho_h}}\right)^{\frac{1}{1+\gamma}}.$$

The steady-state aggregate output has the following representation:

$$Y^{\eta+\sum\lambda_i-\eta\underline{\alpha}_K}\left(\frac{\delta+\rho_h}{\overline{\alpha}_K}\right)^{\eta\underline{\alpha}_K}=\left[\zeta\left(\prod_{i=1}^{N}A_i^{\lambda_i}\right)\left(\frac{\overline{\alpha}_L}{\psi}\frac{1}{(\overline{\alpha}_K+\overline{\alpha}_L)-\frac{\delta\overline{\alpha}_K}{\delta+\rho_h}}\right)^{\frac{\underline{\alpha}_L}{1+\gamma}}\right]^{\eta}\prod_{i=1}^{N}\left[\frac{\Theta_i}{\rho_e(\eta-1)}\left(Y-\frac{\eta\Phi_i}{\widetilde{\lambda}_i}\right)\right]^{\lambda_i}.$$

On the other hand, Y is bounded from below, i.e., $Y\geq Y_{min}$, because the cutoff productivity is bounded from below, i.e., $Z_i^*\geq\underline{Z}_i$ (when $\Theta_i=0$, all firms produce, and this includes the most unproductive firms). The lower bound of output Y_{min} is given by

$$Y=\zeta\prod_{i=1}^{N}A_i^{\lambda^i}L^{\underline{\alpha}_L}\prod_{i=1}^{N}\left[\frac{Z_i^*}{\underline{Z}_i}\right]^{\lambda_i}K^{\underline{\alpha}_K}\geq\left[\zeta\prod_{i=1}^{N}A_i^{\lambda^i}L^{\underline{\alpha}_L}\right]K^{\underline{\alpha}_K}\equiv Y_{min}.$$

This section studies the steady states of the model. Suppose all sectors share the same level of with-in sector heterogeneity: $\eta_i=\eta$, then in the steady state, the aggregate output solves following nonlinear equation

$$Y^{\eta+\sum\lambda_i-\eta\underline{\alpha}_K}\left(\frac{Y}{K}\right)^{\eta\underline{\alpha}_K}=\left(\zeta\prod_{i=1}^{N}A_i^{\lambda_i}L^{\underline{\alpha}_L}\right)^{\eta}\prod_{i=1}^{N}\left[\frac{\Theta_i}{\rho_e(\eta-1)}\left(Y-\frac{\eta\Phi_i}{\widetilde{\lambda}_i}\right)\right]^{\lambda_i},\tag{C.1}$$

where

$$\zeta=\prod_{i=1}^{N}\left[k_i^{\alpha_{K,i}}l_i^{\alpha_{L,i}}\prod_{j=1}^{N}\left(\widetilde{\alpha}_{S,ij}\frac{\widetilde{\lambda}_i}{\widetilde{\lambda}_j}\right)^{\alpha_{S,ij}}\right]^{\lambda_i}\cdot\prod_{i=1}^{N}\left(\frac{\varphi_i}{\widetilde{\lambda}_i}\right)^{\varphi_i},\quad\underline{\alpha}_K=\boldsymbol{\lambda}'\boldsymbol{\alpha}_K,\quad\underline{\alpha}_L=\boldsymbol{\lambda}'\boldsymbol{\alpha}_L.$$

Figure 8 shows how steady-state output is determined.[13] The blue solid curve plots the left-hand side of equation (C.1), while the red dashed curve plots the right-hand side. The intersection of two curves yields two steady states Y_L, Y_H. However, here we will mainly focus on the high steady state Y_H.

[13]Parameters: $\Phi=0.005$, $\boldsymbol{\varphi}=\begin{bmatrix}0.5\\0.5\end{bmatrix}$, $\alpha_M=0.5$, $\eta=6$, $\boldsymbol{\alpha}_S=\alpha_M\cdot\begin{bmatrix}a & 1-a\\1-b & b\end{bmatrix}$, $a=0.3$, $b=0.3$, $\gamma=0$, $\psi=1$, $\delta=0.025$.

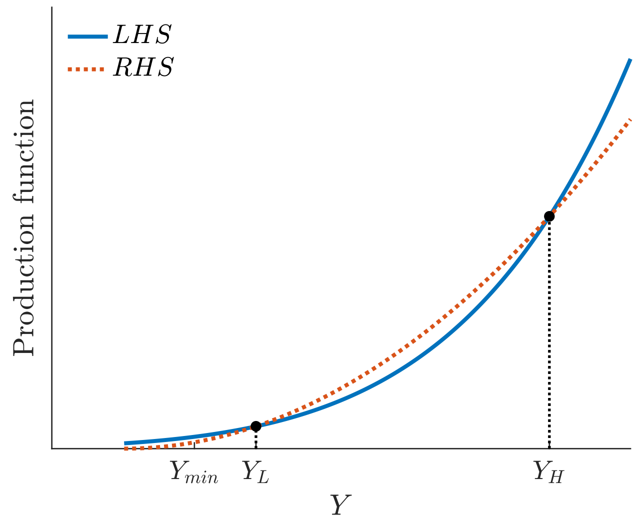

Figure 8: Steady-state aggregate output

C. 2 Admissible Parameters

In this section we find endogenous restrictions on (Φ, Θ) combination such that interior solution of steady state output exists. To make our analysis precise and simple, we assume symmetry in fixed cost Φ across sectors, i.e. $\Phi_i = \Phi$. For simplicity, we also assume all sectors share same productivity distribution, i.e. $\eta_i = \eta$. Consequently $\underline{Z}_i = \underline{Z}$.

In the steady state, the aggregate production function is

$$Y = \zeta \prod_{i=1}^{N} \left[A_i \frac{Z_i^*}{\underline{Z}_i} \right]^{\lambda_i} K^{\underline{\alpha}_K} L^{\underline{\alpha}_L} = \zeta \prod_{i=1}^{N} \left[A_i^{\lambda_i} \left[\frac{\Theta_i}{\rho_e(\eta - 1)} \left(1 - \frac{\eta \Phi_i}{\widetilde{\lambda}_i Y} \right) \right]^{\lambda_i/\eta} \right] K^{\underline{\alpha}_K} L^{\underline{\alpha}_L},$$

which is equivalent to

$$Y^{1+\sum_{i=1}^{N} \lambda_i/\eta} = \zeta \left[\prod_{i=1}^{N} A_i^{\lambda_i} \left[\frac{\Theta_i}{\rho_e(\eta - 1)} \left(Y - \frac{\eta \Phi_i}{\widetilde{\lambda}_i} \right) \right]^{\lambda_i/\eta} \right] K^{\underline{\alpha}_K} L^{\underline{\alpha}_L}.$$

Denote $\kappa = 1 + \sum_{i=1}^{N} \lambda_i/\eta$. Then the system of steady state can be expressed as the following

$$Y = \left\{ \zeta \left[\prod_{i=1}^{N} A_i^{\lambda_i} \left[\frac{\Theta_i}{\rho_e(\eta - 1)} \left(Y - \frac{\eta \Phi_i}{\widetilde{\lambda}_i} \right) \right]^{\lambda_i/\eta} \right] K^{\underline{\alpha}_K} L^{\underline{\alpha}_L} \right\}^{\frac{1}{\kappa}}, \tag{C.2}$$

$$\overline{\alpha}_L \frac{Y/K}{C_h/K} = \psi L^{\gamma+1},$$

$$0 = \overline{\alpha}_K \frac{Y}{K} - \delta - \rho_h, \tag{C.3}$$

$$0 = -\delta + (\overline{\alpha}_K + \overline{\alpha}_L) \frac{Y}{K} - \frac{C_h}{K}.$$

From (C.3), we know $Y/K = \frac{\delta+\rho_h}{\overline{\alpha}_K} \equiv Y_K$. Then (C.2) can be simplified as

$$Y = \left[\zeta \prod_{i=1}^{N} A_i^{\lambda_i} L^{\underline{\alpha}_L}\right] K^{\underline{\alpha}_K} \prod_{i=1}^{N} \left[\frac{\Theta_i}{\rho_e(\eta-1)}\left(1-\frac{\eta\Phi_i}{\widetilde{\lambda}_i Y}\right)\right]^{\lambda_i/\eta},$$

$$Y^{1-\underline{\alpha}_K} Y_K^{\underline{\alpha}_K} = \left[\zeta \prod_{i=1}^{N} A_i^{\lambda_i} L^{\underline{\alpha}_L}\right] \prod_{i=1}^{N} \left[\frac{\Theta_i}{\rho_e(\eta-1)}\left(1-\frac{\eta\phi_i}{\widetilde{\lambda}_i}\right)\right]^{\lambda_i/\eta},$$

$$Y^{\eta+\sum\lambda_i-\eta\underline{\alpha}_K} Y_K^{\eta\underline{\alpha}_K} = \left(\zeta \prod_{i=1}^{N} A_i^{\lambda_i} L^{\underline{\alpha}_L}\right)^{\eta} \prod_{i=1}^{N} \left[\frac{\Theta_i}{\rho_e(\eta-1)}\left(Y-\frac{\eta\Phi_i}{\widetilde{\lambda}_i}\right)\right]^{\lambda_i}, \tag{C.4}$$

where

$$\zeta = \prod_{i=1}^{N} \left[k_i^{\alpha_{K,i}} l_i^{\alpha_{L,i}} \prod_{j=1}^{N} \left(\widetilde{\alpha}_{S,ij} \frac{\widetilde{\lambda}_i}{\widetilde{\lambda}_j}\right)^{\alpha_{S,ij}}\right]^{\lambda_i} \cdot \prod_{i=1}^{N} \left(\frac{\varphi_i}{\widetilde{\lambda}_i}\right)^{\varphi_i}, \quad \underline{\alpha}_K = \boldsymbol{\lambda}'\boldsymbol{\alpha}_K, \quad \underline{\alpha}_L = \boldsymbol{\lambda}'\boldsymbol{\alpha}_L.$$

On the other hand Y has a lower bound Y_{min}, because $Z_i^* \geq \underline{Z}_i$. (When $\Theta = 0$, all firms have to produce, this includes the most unproductive firms.) Y_{min} is then given by

$$\begin{aligned} Y &= \zeta \prod_{i=1}^{N} \left[A_i \frac{Z_i^*}{\underline{Z}_i}\right]^{\lambda_i} K^{\underline{\alpha}_K} L^{\underline{\alpha}_L} = \zeta \prod_{i=1}^{N} A_i^{\lambda^i} L^{\underline{\alpha}_L} \prod_{i=1}^{N} \left[\frac{Z_i^*}{\underline{Z}_i}\right]^{\lambda_i} K^{\underline{\alpha}_K} \\ &\geq \zeta \prod_{i=1}^{N} A_i^{\lambda^i} L^{\underline{\alpha}_L} \prod_{i=1}^{N} \left[\frac{\underline{Z}_i}{\underline{Z}_i}\right]^{\lambda_i} K^{\underline{\alpha}_K} = \left[\zeta \prod_{i=1}^{N} A_i^{\lambda^i} L^{\underline{\alpha}_L}\right] K^{\underline{\alpha}_K} \equiv Y_{min}, \end{aligned}$$

$$Y_{min} \equiv \left[\zeta \prod_{i=1}^{N} A_i^{\lambda_i} L^{\underline{\alpha}_L}\right] K^{\underline{\alpha}_K},$$

$$Y_{min} = \left[\left(\zeta \prod_{i=1}^{N} A_i^{\lambda_i} L^{\underline{\alpha}_L}\right) Y_K^{-\underline{\alpha}_K}\right]^{\frac{1}{1-\underline{\alpha}_K}}.$$

Another constraint is to make sure (C.4) is well defined,

$$Y > \frac{\eta\Phi_i}{\widetilde{\lambda}_i}.$$

which places an upper bound on Φ_i,

$$\Phi_i < \frac{Y_{\min} \min(\widetilde{\lambda}_i)}{\eta}.$$

C. 3 Global Dynamics

This section moves from the characterization of the local dynamics around steady states to the global dynamics. Under Pareto distribution, the dynamic system on $\{Y_t, K_t, L_t, I_t, C_{h,t}, W_t, R_t\}$ is given by,

$$Y_t = \zeta \prod_{i=1}^{N} \left[A_{it}^{\lambda_i} \left[\frac{\Theta_{it}}{\rho_e(\eta_i - 1)} \left(1 - \frac{\eta_i \phi_i}{\widetilde{\lambda}_i}\right) \right]^{\lambda_i/\eta_i} \right] K_t^{\alpha_K} L_t^{\alpha_L} \tag{C.5}$$

$$W_t = \overline{\alpha}_L \frac{Y_t}{L_t}, \quad \frac{W_t}{C_{h,t}} = \psi L_t^{\gamma}, \quad R_t = \overline{\alpha}_K \frac{Y_t}{K_t},$$

$$\frac{\dot{C}_{h,t}}{C_{h,t}} = R_t - \delta - \rho_h, \tag{C.6}$$

$$\dot{K}_t = -\delta K_t + (\overline{\alpha}_K + \overline{\alpha}_L) Y_t - C_{h,t}. \tag{C.7}$$

Finally, the clearing condition of the labor markets implies that

$$L_t = \left(\frac{\overline{\alpha}_L}{\psi} \frac{1}{C_{h,t}/Y_t} \right)^{\frac{1}{1+\gamma}}. \tag{C.8}$$

Altogether, we have four variables $\{K_t, Y_t, C_{h,t}, L_t\}$ with four equations (C.5), (C.6), (C.7), and (C.8). Substituting (C.8) into (C.5) yields an analytic formulation of $C_{h,t}$ as a function of (K_t, Y_t):

$$C_h = \left[\zeta \left(\prod_{i=1}^{N} A_{it}^{\lambda_i} \left(\frac{\Theta_i}{\rho_e(\eta_i - 1)} \left(1 - \frac{\eta_i \Phi_i}{\widetilde{\lambda}_i Y_t}\right) \right)^{\frac{\lambda_i}{\eta_i}} \right) K_t^{\alpha_K} \left(\frac{\overline{\alpha}_L}{\psi} Y_t \right)^{\frac{\alpha_L}{1+\gamma}} Y_t^{-1} \right]^{\frac{1+\gamma}{\alpha_L}}.$$

Taking log on both sides yields

$$\begin{aligned} \ln C_{h,t} = \frac{1+\gamma}{\alpha_L} \Bigg[& \ln \zeta + \sum_{i=1}^{N} \left(\lambda_i \ln A_{it} + \frac{\lambda_i}{\eta} \ln \frac{\Theta_i}{\rho_e(\eta_i - 1)} \right) + \frac{\alpha_L}{1+\gamma} \ln \frac{\overline{\alpha}_L}{\psi} \\ & + \sum_{i=1}^{N} \frac{\lambda_i}{\eta_i} \ln \left(1 - \frac{\eta_i \Phi_i}{\widetilde{\lambda}_i Y_t}\right) + \alpha_K \ln K_t + \left(\frac{\alpha_L}{1+\gamma} - 1 \right) \ln Y_t \Bigg]. \end{aligned}$$

We set $A_{it} = A_i$ to consider deterministic case. Taking derivative of both sides with respect to t yields

$$\begin{aligned} \frac{\dot{C}_{h,t}}{C_{h,t}} &= \frac{1+\gamma}{\alpha_L} \left[\sum_{i=1}^{N} \frac{\lambda_i}{\eta_i} \left(\frac{\frac{\eta_i \Phi_i}{\widetilde{\lambda}_i Y_t}}{1 - \frac{\eta_i \Phi_i}{\widetilde{\lambda}_i Y_t}} \right) \frac{\dot{Y}_t}{Y_t} + \alpha_K \frac{\dot{K}_t}{K_t} + \left(\frac{\alpha_L}{1+\gamma} - 1 \right) \frac{\dot{Y}_t}{Y_t} \right] \\ &= \frac{1+\gamma}{\alpha_L} \left[\sum_{i=1}^{N} \frac{\lambda_i}{\eta_i} \left(\frac{\frac{\eta_i \Phi_i}{\widetilde{\lambda}_i Y_t}}{1 - \frac{\eta_i \Phi_i}{\widetilde{\lambda}_i Y_t}} \right) + \frac{\alpha_L}{1+\gamma} - 1 \right] \frac{\dot{Y}_t}{Y_t} + \frac{\alpha_K (1+\gamma)}{\alpha_L} \frac{\dot{K}_t}{K_t}. \end{aligned} \tag{C.9}$$

Substituting (C.9) into (C.6), we have

$$\frac{1+\gamma}{\alpha_L} \left[\sum_{i=1}^{N} \frac{\lambda_i}{\eta_i} \left(\frac{\frac{\eta_i \Phi_i}{\widetilde{\lambda}_i Y_t}}{1 - \frac{\eta_i \Phi_i}{\widetilde{\lambda}_i Y_t}} \right) + \frac{\alpha_L}{1+\gamma} - 1 \right] \frac{\dot{Y}_t}{Y_t} + \frac{\alpha_K (1+\gamma)}{\alpha_L} \frac{\dot{K}_t}{K_t} = \overline{\alpha}_k \frac{Y_t}{K_t} - \delta - \rho_h.$$

Consequently, these yield two differential equations on (K_t, Y_t). That is, we can simplify the dynamic system into a 2-dimensional autonomous dynamical system on (K_t, Y_t).

$$\dot{K}_t = -\delta K_t + (\bar{\alpha}_k + \bar{\alpha}_l) Y_t - C_h (K_t, Y_t),$$
$$\dot{Y}_t = \frac{\left[\overline{\alpha}_K - \frac{\alpha_K(1+\gamma)}{\alpha_L}(\overline{\alpha}_K + \overline{\alpha}_L)\right] Y_t - \left[\delta + \rho_h - \delta \frac{\alpha_K(1+\gamma)}{\alpha_L}\right] K_t + \frac{\alpha_K(1+\gamma)}{\alpha_L} C_h (Y_t, K_t)}{\frac{1+\gamma}{\alpha_L}\left[\sum_{i=1}^{N} \frac{\lambda_i}{\eta_i}\left(\frac{\frac{\eta_i \Phi_i}{\bar{\lambda}_i Y_t}}{1-\frac{\eta_i \Phi_i}{\bar{\lambda}_i Y_t}}\right) + \frac{\alpha_L}{1+\gamma} - 1\right]} \cdot \frac{Y_t}{K_t}.$$

D. Data and Quantitative Exercise

KLEMS. The KLEMS data are from BEA. This data includes capital expenditure, labor expenditure, gross output and value added accounts at industry level. The capital expenditure is computed through aggregating capital_art compensation, capital_IT compensation, capital_ohter compensation, capital_R&D compensation and capital_software compensation accounts. The labor expenditure is computed through aggregating the labor_nocol compensation and labor_col compensation accounts. The KLEMS data is for 63 industries and we can aggregate them into 15 sectors according to NAICS classification. Table 2 and 3 list that each of 63 industries belongs to one of 15 sectors. The industry level capital share and labor share are computed through the following equations:

$$\widetilde{\alpha}_{K,i} = \frac{\texttt{capital expenditure}_i}{\texttt{gross output}_i}, \quad \widetilde{\alpha}_{L,i} = \frac{\texttt{labor expenditure}_i}{\texttt{gross output}_i}$$

Input-output account. The input-output data are from BEA input-output accounts's use table. These data contains information on the uses of commodities j by intermediate i and final users. For BEA aggregation in I-O tables, 15 sectors are approximately the North American Industry Classification System (NAICS) sectors. The industry level intermediate input share is computed through the following equation:

$$\widetilde{\alpha}_{S,ij} = \frac{i\texttt{'s expenditure on commodity } j}{\texttt{gross output}_i}$$

The final expenditure share is computed through the following equation:

$$\varphi_i = \frac{\texttt{final user's expenditure on commoditry } i}{\texttt{total final user expenditure}}$$

These data also contains information on industry level gross operating surplus. We can use this piece of information to calculate the wedge $\mathbb{E}\left[\frac{Z_i}{Z_i^*}|Z_i \geq Z_i^*\right]$.

$$\mathbb{E}\left[\frac{Z_i}{Z_i^*}|Z_i \geq Z_i^*\right] = \frac{P_i O_i}{WL_i + RK_i + \sum P_j S_{ij}} = \frac{\texttt{Sales}_i}{\texttt{Expenditure}_i} = \frac{\texttt{gross output}_i}{\texttt{gross output}_i - \texttt{gross operating surplus}_i} \geq 1$$

With this piece of information, we can also calculate the cost based input-output matrix:

$$\alpha_{K,i} = \frac{\texttt{capital expenditure}_i}{\texttt{gross output}_i - \texttt{gross operating surplus}_i}, \quad \alpha_{L,i} = \frac{\texttt{labor expenditure}_i}{\texttt{gross output}_i - \texttt{gross operating surplus}_i}$$

Industry Name	NACIS-2 digit	NACIS Sector
Farms	1	11
Forestry, fishing, and related activities	2	11
Oil and gas extraction	3	21
Mining, except oil and gas	4	21
Support activities for mining	5	21
Utilities	6	22
Construction	7	23
Wood products	8	31G
Nonmetallic mineral products	9	31G
Primary metals	10	31G
Fabricated metal products	11	31G
Machinery	12	31G
Computer and electronic products	13	31G
Electrical equipment, appliances, and components	14	31G
Motor vehicles, bodies and trailers, and parts	15	31G
Other transportation equipment	16	31G
Furniture and related products	17	31G
Miscellaneous manufacturing	18	31G
Food and beverage and tobacco products	19	31G
Textile mills and textile product mills	20	31G
Apparel and leather and allied products	21	31G
Paper products	22	31G
Printing and related support activities	23	31G
Petroleum and coal products	24	31G
Chemical products	25	31G
Plastics and rubber products	26	31G
Wholesale trade	27	42
Retail trade	28	44RT
Air transportation	29	48TW
Rail transportation	30	48TW
Water transportation	31	48TW

Table 2: Merge 63 industries into 15 sectors(1)

Industry Name	NACIS-2 digit	NACIS Sector
Truck transportation	32	48TW
Transit and ground passenger transportation	33	48TW
Pipeline transportation	34	48TW
Other transportation and support activities	35	48TW
Warehousing and storage	36	48TW
Publishing industries, except internet (includes software)	37	51
Motion picture and sound recording industries	38	51
Broadcasting and telecommunications	39	51
Data processing, internet publishing, and other information services	40	51
Federal Reserve banks, credit intermediation, and related activities	41	FIRE
Securities, commodity contracts, and investments	42	FIRE
Insurance carriers and related activities	43	FIRE
Funds, trusts, and other financial vehicles	44	FIRE
Real estate	45	FIRE
Rental and leasing services and lessors of intangible assets	46	FIRE
Legal services	47	PROF
Computer systems design and related services	48	PROF
Miscellaneous professional, scientific, and technical services	49	PROF
Management of companies and enterprises	50	PROF
Administrative and support services	51	PROF
Waste management and remediation services	52	PROF
Educational services	53	6
Ambulatory health care services	54	6
Hospitals and nursing and residential care facilities	55	6
Social assistance	56	6
Performing arts, spectator sports, museums, and related activities	57	7
Amusements, gambling, and recreation industries	58	7
Accommodation	59	7
Food services and drinking places	60	7
Other services, except government	61	81
Federal	62	G
State and local	63	G

Table 3: Merge 63 industries into 15 sectors (2)

$$\alpha_{S,ij} = \frac{i\text{'s expenditure on commodity } j}{\text{gross output}_i - \text{gross operating surplus}_i}$$

In addition, since we assume that the firm level productivity follows the Pareto distribution, the industry level productivity wedge is related to Pareto distribution parameter in the following way:

$$\mathbb{E}\left[\frac{Z_i}{Z_i^*}|Z_i \geq Z_i^*\right] = \frac{\eta_i}{\eta_i - 1}$$

Then we can back out distribution parameter η_i through the following equation:

$$\eta_i = \frac{1}{1 - \frac{1}{\mathbb{E}\left[\frac{Z_i}{Z_i^*}|Z_i \geq Z_i^*\right]}}.$$

Debt ratio. We estimate the industry level financial constraint θ_{it} using the industry level debt ratio. The industry level debt ratio data is from the Compustat dataset on Wharton Research Data Services. This database includes debt in current liability (code: dlc), long-term debt (code: dltt), and total asset (code: at) accounts. The firm level debt ratio is computed through the following equation:

$$\texttt{debt ratio} = \frac{\texttt{dlc} + \texttt{dltt}}{\texttt{at}}$$

We use the ratio of each company's sale to corresponding industry's sales to create a weight to modulate the industry level debt ratio.

Fixed cost. The fixed cost can be computed using the following equation:

$$\frac{\Theta_{it}}{\rho_e}\left[\xi\left(Z_{it}^*\right) - \phi_{it}\right] = \frac{\widetilde{\lambda}_{it}}{\mathbb{E}\left[\frac{Z_i}{Z_i^*}|Z_i \geq Z_i^*\right]}$$

where the operational profit is given by

$$\xi(Z_{it}^*) = \widetilde{\lambda}_{it}\left(1 - \widetilde{\alpha}_{K,it} - \widetilde{\alpha}_{L,it} - \sum_{j=1}^{N}\widetilde{\alpha}_{S,ijt}\right)$$

Then we can infer the fixed cost,

$$\phi_{it} = \xi\left(Z_{it}^*\right) - \frac{\widetilde{\lambda}_{it}}{\mathbb{E}\left[\frac{Z_i}{Z_i^*}|Z_i \geq Z_i^*\right]}\frac{\rho_e}{\theta_{it}}$$

Evaluate the likelihood of self-fulfilling business cycles. Then we can calculate the industry level financial multiplier and aggregate financial multiplier:

$$v_i = \left(\frac{\widetilde{\lambda}_i}{\phi_i} - \eta_i\right)^{-1}, \quad \varrho = \sum_{i=1}^{N} v_i\lambda_i, \quad \mu = \frac{1}{1-\varrho}$$

We next evaluate whether μ lies within the indeterminacy region. The necessary and sufficient condition for equilibrium indeterminacy is given by

$$\max\left\{\mu_1^*, \mu_2^*\right\} < \mu < \mu_3^*,$$

$$\mu_1^* = \frac{1+\gamma}{\underline{\alpha}_L}, \quad \mu_2^* = \frac{\delta\left(1+\gamma\right)}{\frac{\overline{\alpha}_K+\overline{\alpha}_L}{\overline{\alpha}_K}\left(\delta+\rho_h\right)\left(1+\gamma\right)\underline{\alpha}_K - \underline{\alpha}_L\rho_h}, \quad \mu_3^* = \frac{1}{\underline{\alpha}_K}.$$

where $\underline{\alpha}_{K,t} = \boldsymbol{\lambda}_t'\boldsymbol{\alpha}_K,\ \underline{\alpha}_{L,t} = \boldsymbol{\lambda}_t'\boldsymbol{\alpha}_L, \overline{\alpha}_{L,t} = \widetilde{\boldsymbol{\lambda}}_t'\widetilde{\boldsymbol{\alpha}}_{L,t},\ \overline{\alpha}_{K,t} = \widetilde{\boldsymbol{\lambda}}_t'\widetilde{\boldsymbol{\alpha}}_{K,t}$.